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PROCEEDINGS
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THE MECHANISM OF RHODOPSIN SYNTHESIS

By Ruth Hubbard and George Wald*

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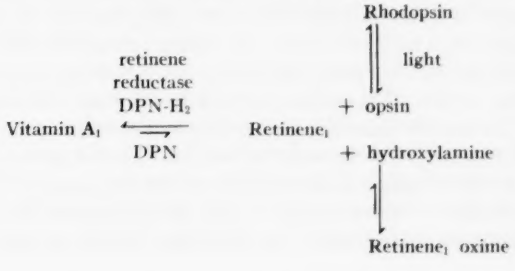
Communicated December 22, 1950

We have recently demonstrated the synthesis of rhodopsin from vitamin A_1 and opsin (rhodopsin-protein) in retinal homogenates and aqueous retinal extracts.[1] This process was shown to be aided by supplementation with homogenates of the pigment layers of the eye—the pigment epithelium and choroid—and with cozymase (DPN). A mechanism was proposed for rhodopsin synthesis, based upon the following considerations.

(1) Vitamin A_1 is in equilibrium with retinene$_1$ through the retinene reductase system, in which cozymase functions as coenzyme.[2] The equilibrium lies far over toward the side of reduction (vitamin A_1), but can be displaced in the oxidative direction by the use of a retinene-trapping reagent such as hydroxylamine, which condenses spontaneously with retinene$_1$ to form retinene$_1$ oxime. In this process the endergonic oxidation of vitamin A_1 to retinene$_1$ is coupled with the exergonic trapping reaction.[1]

(2) Retinene$_1$ condenses spontaneously also with opsin to form rhodopsin.[3] This exergonic process can serve as a retinene-trapping reaction in the retina. Rhodopsin may therefore be synthesized by the oxidation of vitamin A_1 to retinene$_1$ by the retinene reductase system, coupled with the condensation of retinene$_1$ with opsin to form rhodopsin.[1]

These reactions are summarized in the following diagram:



In this system the oxidation of vitamin A_1 to retinene$_1$ is the limiting process. It is presumably here that DPN acts in promoting rhodopsin synthesis.

In the experiments to be described we have examined systematically the effects on rhodopsin synthesis of factors which promote the oxidation of vitamin A_1. We have analyzed in part the contribution of the pigment epithelium. Finally, we have assembled a minimal system of well-defined components, which synthesizes rhodopsin from vitamin A_1 in aqueous solution.

Procedures.—The retinas used in these experiments were dissected from the eyes of dark adapted frogs (*Rana pipiens*) under dim red light. The pigment layers of the eye—the combined pigment epithelium and choroid layer—were collected separately. The retinas were bleached by exposure to an intense tungsten source, filtered through Corning glasses 3387 and 3966 to remove heat, and violet and near ultraviolet radiation which might otherwise destroy vitamin A_1 and retinene$_1$. The retinas were left for 1–$1^1/_2$ hours in the light at room temperature, to allow all reactions to be completed. They were then colorless, all the retinene$_1$ liberated in the bleaching of rhodopsin having been reduced to vitamin A_1.

Preparations for the synthesis of rhodopsin were incubated in the dark at room temperature, usually overnight. Rhodopsin was extracted with 2 per cent crystalline digitonin in water, the digitonin acting as solubilizer. Digitonin was used also to bring vitamin A_1 or retinene$_1$ into aqueous solution.

Digitonin extracts were examined for the presence of rhodopsin by measuring the absorption spectrum in darkness, bleaching in bright tungsten light filtered through yellow Corning glass 3384, and re-measuring the absorption spectrum. The difference in absorption before and after bleaching —the so-called difference spectrum—identifies rhodopsin and measures its quantity. All the figures in the present paper are such difference spectra.

Opsin was prepared by methods similar to those used to make the purest preparations of rhodopsin.[3] Cattle retinas were bleached to colorlessness, disintegrated mechanically, and the outer segments of the rods were isolated by centrifuging in 35 per cent sucrose solution. The rods remain in suspension, while the remaining retinal tissues are sedimented. The suspension in sugar solution is diluted, and the rods collected by centrifuging. They are tanned in 4 per cent alum, and washed repeatedly with water and neutral phosphate buffer. Sometimes they are lyophilized, and extracted with petroleum ether. The residue of these treatments is free from water-soluble and fat-soluble material, and most of its remaining proteins have been rendered insoluble by the alum treatment. From this residue opsin is extracted into clear, colorless solution with 2 per cent aqueous digitonin.

Oxidative Factors: "Succinoxidase."—We have proposed that cozymase stimulates rhodopsin production by acting in the retinene reductase system,

in which it oxidizes vitamin A_1 to retinene$_1$, and is itself reduced in the process. If this view is correct, an enzyme system which re-oxidizes cozymase should aid in rhodopsin synthesis.

The so-called "succinoxidase" of heart muscle is of this nature. It is a particulate complex that includes all the components needed to transfer hydrogen from cozymase to oxygen: cytochrome oxidase, the cytochromes, and riboflavine enzymes. In our experiments we have used the beef heart

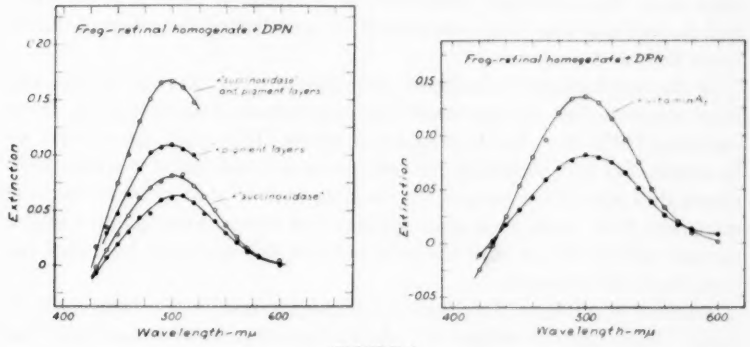


FIGURE 1

Stimulation of rhodopsin synthesis with "succinoxidase" and pigment layer homogenate. Frog retinal homogenate supplemented with DPN, and incubated alone, with "succinoxidase," with pigment layer homogenate, and with both "succinoxidase" and pigment layer homogenate. Difference spectra of rhodopsin extracted after incubation are shown. Each preparation contained 4.5 retinas, 1.5 mg. DPN, and 0.002 M NaF in a final volume of 1.0 ml. Each portion of pigment layer homogenate contained pigment layers of 9 eyes. "Succinoxidase" was supplemented with cytochrome *c*, 1.5×10^{-5} M.

FIGURE 2

Stimulation of rhodopsin synthesis by vitamin A_1. Frog retinal homogenate supplemented with DPN, and incubated alone and with added vitamin A_1. Difference spectra of rhodopsin extracted after incubation are shown.

preparation of Ball and Cooper,[4] supplemented with cytochrome *c* in a final concentration of 10^{-5} *M*.

Figure 1 shows the results of one such experiment. Retinas of 9 frogs were isolated and bleached to colorlessness (vitamin A_1 and opsin). The retinas were homogenized, supplemented with cozymase and divided into 4 equal portions. These were incubated in the dark at room temperature in the following way: one diluted only with buffer, the second with "succinoxidase," the third with a homogenate of the pigment layers from 9 eyes and the fourth with both "succinoxidase" and the same amount of pigment

layer homogenate. The final volume of each mixture was 1.0 ml. After incubation, all preparations were centrifuged, and the solid residues were extracted with 0.65 ml. of 2 per cent aqueous digitonin.

Figure 1 contains the difference spectra of these extracts. It shows that the addition of "succinoxidase" to the retinal homogenate increased the yield of rhodopsin 33 per cent, or 50 per cent in the presence of pigment layer homogenate. The latter roughly doubled the yield of rhodopsin, whether or not "succinoxidase" had also been added. These observations show that "succinoxidase" promotes the synthesis of rhodopsin from vitamin A_1, and that this effect is relatively independent of the action of the pigment layers.

In the mechanism of rhodopsin synthesis that we have proposed, such respiratory factors as "succinoxidase" participate only indirectly, by reoxidizing DPN-H_2. In the presence of excess DPN, such factors may not be required at all. Probably for this reason we have found in some experiments that removal of oxygen or poisoning with 10^{-3} *M* cyanide interferes relatively little with rhodopsin synthesis in retinal homogenates supplemented with DPN, or with the promotion of this synthesis by added pigment layer homogenate.

Addition of Vitamin A_1.—Figure 2 shows the results of a typical experiment. Right and left retinas of 5 dark adapted grass frogs were dissected into separate vessels and were bleached to colorlessness. One set of retinas was homogenized with 0.45 ml. neutral phosphate buffer and one drop of cottonseed oil containing 0.5 per cent of the antioxidant, α-tocopherol; the other set with an identical mixture in which the oil contained also 0.07 mg. vitamin A_1. To each homogenate, 1.8 mg. DPN was added, and both were incubated for 12 hours in the dark. They were centrifuged, and rhodopsin was extracted from the solid residues with 0.65 ml. of 2 per cent aqueous digitonin. The difference spectra of these extracts are shown in figure 2. The added vitamin A_1 had increased the synthesis of rhodopsin about 65 per cent.

We have found that a more effective way to add vitamin A_1 to such a system is to grind the oil containing the vitamin with a dry powder of lyophilized retinas. The powder is afterward suspended in buffer solution, and is incubated in the dark. Such an experiment is described below (Fig. 4).

Role of the Pigment Epithelium.—We have shown that the addition of pigment layer homogenate greatly increases the yield of rhodopsin in retinal homogenates, whether otherwise unsupplemented, or supplemented with DPN or "succinoxidase."

It might be supposed that the pigment layers themselves synthesize rhodopsin. We have, however, supplemented pigment layer homogenate strongly with vitamin A_1 and DPN, and could observe only a trace of rhodopsin formation, no more than might be explained by contamination

with a few rods. Pigment layers have also been tested for the presence of retinene reductase by incubating with DPN and hydroxylamine, the latter to act as a retinene-trapping reagent; no evidence was obtained of oxidation of vitamin A_1 to retinene$_1$. The pigment layers apparently do not form rhodopsin; they seem to lack opsin, and probably also retinene reductase.

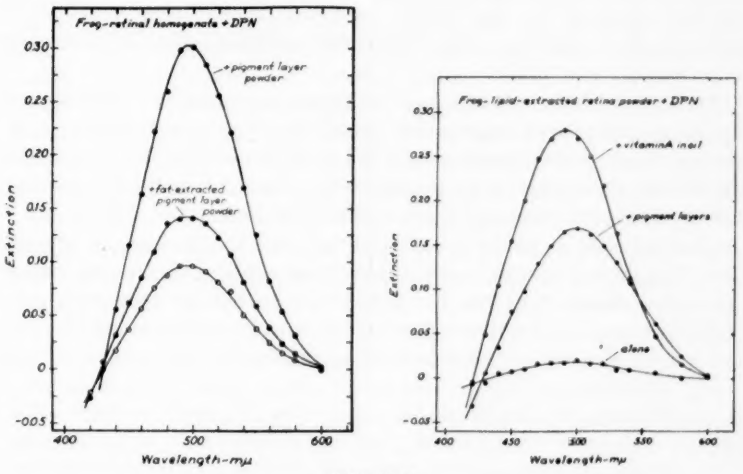


FIGURE 3

The fat-soluble factor of the pigment layers. Frog retinal homogenate supplemented with DPN was incubated alone, with a brei of lyophilized pigment layers, and with a similar brei of lyophilized pigment layers which had previously been extracted with petroleum ether. Difference spectra of rhodopsin extracted after incubation are shown. The fat-extracted pigment layers had lost most of their capacity to stimulate rhodopsin synthesis.

FIGURE 4

Synthesis of rhodopsin from extra-retinal vitamin A_1. Lyophilized retinas were extracted with petroleum ether to remove vitamin A_1, and were supplemented with DPN in neutral phosphate buffer. Equal portions were incubated alone, with lyophilized pigment layers, and with vitamin A_1 in oil. The fat-extracted retinas make almost no rhodopsin. The rhodopsin formed in the other preparations is derived therefore from the vitamin A_1 added in oil or from the pigment layers. About 5 times as much vitamin A_1 was added in oil as was contained in the pigment layers.

The effectiveness of the pigment layers in our experiments is due principally to a fat-soluble factor. If pigment layers are extracted with petroleum ether, they lose most of their capacity to stimulate rhodopsin synthesis. This is shown in figure 3.

A retinal homogenate which had been supplemented with DPN was incubated in the dark (*a*) alone; (*b*) with 6.3 mg. of a powder of lyophilized pigment layers; and (*c*) with 6.3 mg. of the same pigment layer powder which had been extracted twice with petroleum ether. After 12 hours incubation, rhodopsin was extracted from all three portions with digitonin solution. Figure 3 shows the difference spectra of these extracts. The addition of lyophilized pigment layers to the retinal tissue had tripled the yield of rhodopsin (*b*); but this effect was very much reduced on extraction with petroleum ether (*c*). The same result is obtained when the extraction is performed at $-15°C$.

The fat-soluble factor of the pigment layers is vitamin A_1. The pigment layers contain a large quantity of vitamin A_1, in this species about 4 μg. per eye, concentrated almost wholly in the pigment epithelium.[5] A simple experiment shows that in our homogenates, vitamin A_1 is transferred from the pigment epithelium to the retinal tissue for rhodopsin synthesis.

Dark adapted retinas of 8 frogs were bleached to colorlessness, lyophilized, ground to a powder, and extracted thoroughly with petroleum ether to remove vitamin A_1. The dry powder was divided into three equal portions, and treated as follows: portion (*a*) was left unsupplemented; (*b*) was stirred with 0.2 ml. cottonseed oil containing 0.27 mg. vitamin A_1 and 1 mg. α-tocopherol; and (*c*) was mixed with a lyophilized powder of 12 pigment layers. Each portion was stirred with 0.33 ml. phosphate buffer, pH 6.8, containing 1.2 mg. DPN, and was incubated for 12 hours in the dark. It was then extracted with 0.65 ml. of 2 per cent digitonin solution, and the difference spectra of these extracts were measured.

Figure 4 shows the results of this experiment. From the very small synthesis of rhodopsin in (*a*), it is clear that extraction of the retinal tissue with petroleum ether had removed almost all vitamin A_1. Therefore virtually all the rhodopsin in (*b*) and (*c*) must have been made from extra-retinal vitamin A_1. In (*b*) the vitamin A_1 was supplied as a solution in oil, and had resulted in a 40 per cent yield of rhodopsin. In (*c*) the vitamin A_1 for rhodopsin synthesis was supplied by the lyophilized pigment layers, and had resulted in a 25 per cent yield. It should be noted that the amount of vitamin A_1 added in oil in (*b*) was about 5 times as great as the pigment layers contained in (*c*).

This experiment shows unequivocally that in homogenates the vitamin A_1 of the pigment epithelium is available for rhodopsin synthesis by the retinal tissue. It shows also that supplementation of retinal tissue with vitamin A_1 can substitute for supplementation with whole pigment layers. It should be recalled that these experiments are all conducted in the presence of excess DPN. Under these circumstances it seems that the primary action of the pigment layers is to supply the retinal tissue with vitamin A_1.

This, however, is not their only contribution to rhodopsin synthesis. The activity of frog pigment layers is greatly reduced by heating for 2–3 minutes at 100°C., though they still contain large amounts of vitamin A_1. From homogenized pigment layers, deep yellow, opalescent solutions can be prepared in neutral M/15 phosphate buffer, which strongly stimulate the synthesis of rhodopsin in retinal homogenates, though they contain very little vitamin A_1. The activity of such solutions is destroyed by boiling for 2–3 minutes. They contain heat-labile factors, probably proteins, which aid in rhodopsin synthesis. Their yellow color is due principally to riboflavine. Their composition and action are being examined further.[6]

It must not be forgotten also that in the intact eye the layer of rods and cones depends primarily upon the choroidal circulation for its exchanges with the blood, and receives oxygen and metabolites from this source by diffusion through the intervening pigment epithelium. In cold-blooded animals, the retina has no blood vessels, and except for the sparse hyaloid vessels, depends entirely upon the choroidal circulation. In mammals, the retinal blood vessels supply primarily the inner retinal layers, and the sensory epithelium still depends principally upon the choroid.[7] These are added reasons why the normal functioning of the rods and the synthesis of rhodopsin *in vivo* demand close contact between the retina and the pigment epithelium.

Synthesis of Rhodopsin in a System of Known Components.—If our assumptions concerning the mechanism of rhodopsin synthesis are valid, it should be possible to perform this synthesis in a system containing only four components: vitamin A_1, retinene reductase, cozymase and opsin.

Of these substances, only retinene reductase is not yet available in highly purified condition. Bliss, however, has reported that the equilibrium between vitamin A_1 and retinene$_1$ is catalyzed by crude preparations of liver alcohol dehydrogenase, with DPN as coenzyme.[8] We have confirmed this observation, using the crystalline alcohol dehydrogenase of horse liver, prepared by Bonnichsen.[9] Dr. Warren Yudkin in our laboratory has found also that a homogenate of frog retinas, supplemented with cozymase, oxidizes ethyl alcohol. It is possible therefore that retinene reductase is identical with liver alcohol dehydrogenase. The relation between these enzymes is being examined further.[10]

We have assembled in aqueous digitonin solution the four components which should constitute a minimal system for the synthesis of rhodopsin: vitamin A_1, concentrated from fish liver oils; crystalline alcohol dehydrogenase from horse liver; cozymase (Sigma, 90 per cent pure); and purified opsin, prepared as described above. As control, an identical mixture was prepared which lacked only the alcohol dehydrogenase.

Figure 5 shows the results of such an experiment. In this instance the control mixture generated a small amount of rhodopsin, the source of which

is not yet clear. In the complete mixture, however, rhodopsin is synthesized with high efficiency. Indeed, the opsin used in this experiment, when mixed with excess retinene$_1$, yielded almost exactly as much rhodopsin as was formed from vitamin A$_1$. It seems therefore that the yield of rhodopsin in this experiment was limited only by the amount of available opsin.

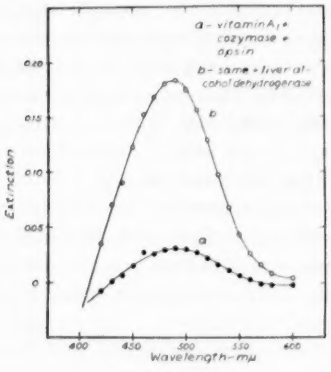


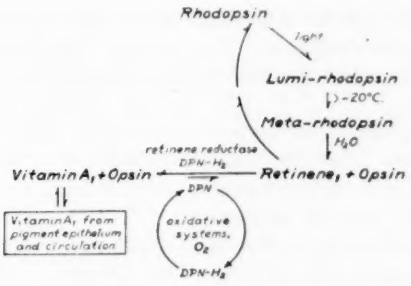
FIGURE 5

Synthesis of rhodopsin in a solution of known components. The upper curve shows the difference spectrum of rhodopsin synthesized by incubating together: vitamin A$_1$, 0.35 mg. in 0.08 ml. cottonseed oil containing 0.5 per cent α-tocopherol; crystalline horse liver alcohol dehydrogenase, 0.6 mg.; cozymase, 0.2 mg.; and frog opsin in 2 per cent digitonin. Final volume, 0.8 ml., in neutral phosphate buffer. The lower curve shows the rhodopsin formed in an identical mixture lacking only the alcohol dehydrogenase. The difference spectra were measured in the presence of hydroxylamine, 0.17 *M*, to block regeneration. The opsin used in this experiment, incubated separately with excess retinene$_1$, yielded rhodopsin with a maximal density of 0.18, almost exactly the same amount as was synthesized here from vitamin A$_1$.

Discussion.—The net result of our experiments is to confirm the view that rhodopsin is synthesized by the oxidation of vitamin A$_1$ to retinene$_1$, coupled with the condensation of retinene$_1$ with opsin. All factors known to promote the oxidation of vitamin A$_1$ aid in the synthesis of rhodopsin: vitamin A$_1$, cozymase and oxidative systems, such as "succinoxidase," which keep cozymase oxidized. The pigment epithelium has been shown to stimulate rhodopsin synthesis in our experiments primarily by supplying one of these factors, vitamin A$_1$, to the retinal tissue. The water-soluble, heat-labile factor of the pigment epithelium is still to be identified, and its function determined. The proposed mechanism, however, is established unequivocally with the demonstration that rhodopsin is synthesized in solution in the system of known components just described.

Is this the only pathway of rhodopsin synthesis? The only reply to this question possible at present is that the mechanism we have described may well be adequate, and that there is as yet no evidence of an alternative mechanism.

The known reactions that compose the rhodopsin system can therefore be formulated as follows:



The right-hand portion of the diagram indicates that all the intermediate stages in bleaching may not be retraced when retinene$_1$ and opsin recombine to form rhodopsin.

Of the four components from which the rhodopsin system has been assembled *in vitro*—vitamin A_1, retinene reductase or alcohol dehydrogenase, cozymase and opsin—only the last appears to be confined to the retina, and indeed to the outer segments of the rods. All the other components have a wider distribution in tissues, and were obtained in our experiments from extra-retinal sources. The specific factor that probably restricts the synthesis of rhodopsin to rods, and limits its quantity there, is opsin.

It is important to note that the rhodopsin system, though isolable *in vitro* as we have described, maintains basic connections *in vivo* with the circulation and the general metabolism. Some years ago it was shown that during light adaptation *in vivo*, the retina loses vitamin A_1 by diffusion into the pigment epithelium and the blood; and that it recaptures vitamin A_1 during dark adaptation by binding it in non-diffusible form in rhodopsin.[11] These interchanges, confirmed and reemphasized in the present experiments, connect the rhodopsin system with the metabolism of vitamin A_1 throughout the organism, and with a more remote circumstance, its nutritional supply.

Through its dependence upon cozymase, the rhodopsin system is connected also with the general processes of retinal respiration and fermentation; and these in turn depend upon the blood circulation for supplies of oxygen and metabolites. The retinal metabolism determines not only the oxidation-reduction state of cozymase, but probably must maintain its concentration by continuous synthesis, for cozymase appears to be under constant attack by cellular nucleotidases.[12] The rhodopsin system, therefore, though it can be isolated in solution with four essential components, makes multiple connections *in vivo* with the circulation and the general metabolism.

Summary.—It was proposed earlier that rhodopsin is synthesized by the oxidation of vitamin A_1 to retinene$_1$ by cozymase (DPN) and retinene

reductase, coupled with the condensation of retinene$_1$ with opsin (rhodopsin-protein) to form rhodopsin. The present experiments confirm this view. There is as yet no evidence for the formation of rhodopsin by another mechanism.

Factors which promote the oxidation of vitamin A_1 by retinal homogenates, all are shown to aid the synthesis of rhodopsin. These include cozymase, vitamin A_1 and oxidative systems such as heart particle "succinoxidase" which keep cozymase oxidized.

The pigment epithelium of the eye is shown to promote rhodopsin synthesis by supplying vitamin A_1 to the retinal tissue. The pigment layers of the eye contain also a heat-labile, water-soluble factor, still unidentified, that stimulates the synthesis of rhodopsin. Through cozymase, the rhodopsin system is connected with the general retinal metabolism, and this in turn depends for supplies of oxygen and metabolites upon exchanges with the blood circulation of the choroid, by diffusion through the intervening pigment epithelium.

A system has been assembled from four well-defined components, which synthesizes rhodopsin from vitamin A_1 in solution. It contains vitamin A_1, cozymase, opsin and crystalline liver alcohol dehydrogenase which substitutes for retinene reductase.

* This research was supported in part by a grant from the Medical Sciences Division of the Office of Naval Research. A preliminary account of most of these experiments was presented at the 18th International Physiological Congress in Copenhagen, August 15–18, 1950 (*Abstracts of Communications*, p. 264).

[1] Wald, G., and Hubbard, R., PROC. NATL. ACAD. SCI., **36,** 92 (1950).

[2] Wald, G., and Hubbard, R., *J. Gen. Physiol.*, **32,** 367 (1948–1949); Wald, G., *Science*, **109,** 482 (1949); *Biochim. et Biophys. Acta*, **4,** 215 (1950).

[3] Wald, G., and Brown, P. K., PROC. NATL. ACAD. SCI., **36,** 84 (1950).

[4] Ball, E. G., and Cooper, O., *J. Biol. Chem.*, **180,** 113 (1949). We are very grateful to these workers for a sample of their "succinoxidase" preparation.

[5] Wald, G., *J. Gen. Physiol.*, **19,** 351, 781 (1935–1936).

[6] Bliss has mentioned in a recent abstract that a heat-labile factor, which he says can be extracted from the pigment layers of the eye with 0.5 *M* NaCl, promotes the regeneration of rhodopsin after bleaching in digitonin solution (Bliss, A. F., *Fed. Proc.*, **9,** 12 (1950)). The process in question apparently does not involve vitamin A_1, and seems to be concerned with the regeneration of rhodopsin from retinene$_1$ and opsin. The connection with the present observations is problematical.

[7] Weiss, O., *Kurzes Handb. d. Ophth.*, **2,** 1 (1932).

[8] Bliss, A. F., *Biol. Bull.*, **97,** 221 (1949).

[9] Bonnichsen, R. K., *Acta Chem. Scand.*, **4,** 715 (1950). We should like to thank Dr. Bonnichsen for a gift of crystalline liver alcohol dehydrogenase, and for his assistance with preliminary experiments, while one of us (R. H.) was a guest at the Carlsberg Laboratory in Copenhagen last summer.

[10] It is noteworthy in this connection that yeast alcohol dehydrogenase, prepared by a modification of Racker's method (Racker, E., *J. Biol. Chem.*, **184,** 313 (1950)) does not appear to catalyze the retinene$_1$-vitamin A_1 equilibrium (R. K. Bonnichsen, personal communication). It is interesting also that monoiodoacetate (0.002 *M*), which is

known to inhibit the yeast enzyme, does not inhibi, either liver alcohol dehydrogenase or retinene reductase.

[11] Wald, G., *J. Gen. Physiol.*, **19**, 781 (1935–1936); *Biol. Symp.*, **7**, 43 (1942).

[12] Mann, P. J. G., and Quastel, J. H., *Biochem. J.*, **35**, 502 (1941); Handler, P., and Klein, J. R., *J. Biol. Chem.*, **143**, 49 (1942); McIlwain, H., and Rodnight, R., *Biochem. J.*, **44**, 470 (1949).

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*POLYPLOIDY IN YEAST AND ITS BEARING ON THE OCCURRENCE OF IRREGULAR GENETIC RATIOS**

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Communicated by G. W. Beadle, December 16, 1950

The significance of irregular ratios, i.e., phenotypic ratios of 4:0, 3:1, 1:3, and 0:4 in asci obtained from crosses which are expected to give 2:2 segregations and commonly do so, has been the subject of considerable speculation in yeast genetics. Lindegren[1] has cited such irregularities as evidence that a gene can be affected by its allele when the two are present in the same cell. On this interpretation, the 4:0 and 3:1 asci are those in which the dominant gene has converted the recessive allele to the dominant condition; the 1:3 and 0:4 cases are indicative of the degradation of the dominant by the recessive allele.

The gene-conversion hypothesis does not lend itself at the present time to critical test, and therefore depends for its validation on the exclusion of other explanations based on known genetic mechanisms. To this end Lindegren and his colleagues have shown that neither multiple factors nor orthodox mutation can provide a general explanation for irregular ratios. Mundkur[2] has demonstrated the inadequacy of the Winge and Roberts[3] hypothesis that accounts for these ratios in terms of an extra mitosis in the formation of ascospores. Finally, Mundkur[4] has excluded polyploidy as a factor in this problem on the grounds that it will not account for all of the irregularities that have been found.

The evidence presented below that polyploidy does in fact occur in yeast has prompted us to reappraise its possible role in the interpretation of irregular ratios.

Evidence of Diploid Ascospores.—Among 64 four-spored linear asci whose spores were tested for mating type, galactose fermentation, and growth habit in liquid medium, one ascus was found which exhibited an irregular ratio for each of these characters. The asci were obtained from a cross between two clones of *Saccharomyces* that were presumably haploid and of composition *agf* and *αGF*, respectively.[5] Clones were grown from the four

spores of the exceptional ascus (No. 55) and were designated A, B, C and D, in the order in which the spores occurred in the linear ascus. Each of the clones gave a negative reaction in mating-type tests with the standard *a* and *α* strains. Clones A, C and D all fermented galactose within 24 hours after being placed in Durham fermentation tubes. Clone B failed to ferment galactose during the 96 hours of the test. All of the clones exhibited the dominant flaky character in liquid medium.

The failure of the four clones to mate with the standards suggested the possibility that each clone consisted of cells heterozygous for the mating-type alleles. Evidence for this view was obtained when it was found that each of the clones proved to be self-sporulating and produced asci on gypsum slants. Twelve asci, three from each of the four clones, were dissected and tests were made for mating type, galactose fermentation, and

TABLE 1

THE RESULTS OF TESTS OF ASCI OBTAINED FROM THE SELF-SPORULATING CLONES 55 A, B, C AND D

CLONE TESTED	ASCUS NO.	COMPOSITION OF SPORES			
55A	1	aGF	αgF	aGf	αgf
	2	aGf	αGF	..	αgF
	3	αgf	aGF	aGf	αgF
55B	1	αgF	agf	agF	αgf
	2	αgF	αgf	agF	agf
	3	αgF	agf	αgf	agF
55C	1	..	αGF	aGf	aGF
	2	αGF	aGf	aGf	αGF
	3	aGf	αGF	aGf	αGF
55D	1	αgf	αgF	aGF	aGf
	2	agF	αGf	aGF	αgf
	3	aGf	αgf	aGF	αgF

flaky. Ten of the twelve had four viable spores and exhibited a 2:2 segregation for both mating type and flaky. In the remaining two asci, only three of the four spores survived, but in these also a segregation for *a* and *α* and for *F* and *f* was observed. With respect to galactose fermentation, the asci from Clones A and D segregated for *G* and *g*, the asci from Clone B gave non-fermenters only, and the asci from Clone C gave fermenters only (table 1).

The results of these tests show that the spores of Ascus 55 had the following composition: Spore A, *aα Gg Ff;* Spore B, *aα gg Ff;* Spore C, *aα GG Ff;* and Spore D, *aα Gg Ff*. Since the three loci gave no evidence of linkage in these experiments, there is good reason to infer a disomic condition for at least three chromosomes of the complement and it is probable that each of the spores was actually diploid. Thus the irregular

ratios which were exhibited in Ascus 55 can be resolved in terms of an orthodox polyploid explanation.

Discussion.—The information at hand does not permit us to distinguish between two alternatives for the origin of Ascus 55. First, the cell from which the ascus arose may have been tetraploid, of composition *aaαα GGgg FFff*. On this basis, the compositions of the four spores of Ascus 55 would be accounted for on the assumption of a normal meiosis in which a crossover between the locus of *G* and the centromere had occurred. The manner of origin of the tetraploid cell itself is suggested by the observations of Lindegren[1] that asci are occasionally found within clones of single mating type. These asci have been interpreted to be the result of "illegitimate" hybridization between cells of like mating type. Such fusions would give a certain proportion of *aa* and *αα* diploids and the mixing of the two clones by Lindegren's method should produce tetraploids of the type suggested above.

Second, the diploid spores may have been the consequence of an extra division in spore formation. Suppose, for example, that a diploid ascogenous cell had produced four nuclei of composition *aGF*, *αGf*, *agF* and *αgf*, respectively, and that each nucleus had undergone a mitosis; the result would be eight nuclei, as follows: (1) *aGF* (2) *aGF*, (3) *αGf*, (4) *αGf*, (5) *agF*, (6) *agF*, (7) *αgf* and (8) *αgf*. The subsequent nuclear fusion of 1×3, 2×7, 4×5 and 6×8 would result in diploid spores of the kind encountered in Ascus 55. The precedent for an extra division is furnished by our own observations and those of Winge and Roberts[3] that asci are occasionally found with more than four viable spores. It should be noted that the fusion of nuclei of like mating type following the extra mitosis would provide diploid mating clones from which tetraploid zygotes could be obtained.

The foregoing discussion suggests the possibility that clones derived from haploid spores may consist of both haploid and diploid cells. A cross between two such clones of opposite mating type should produce, in addition to diploid and tetraploid zygotes, a number of triploid zygotes. The relative proportions of the three classes would depend on their relative viabilities and on the time and frequency of diploidization within the parent clones. As indicated above, tetraploid clones could also be the result of the fusion of two diploid spores derived from the extra mitosis and triploid clones would be obtainable from these in further crosses. Spores obtained from the tetraploid and triploid lines should in turn give rise to aneuploid clones of various kinds.

These polyploid situations make possible an array of genetic combinations that can furnish material from which all of the types of irregular ratios can be obtained. Consider, for example, some of the zygote combinations that could be expected from a cross between a galactose-fermenter clone and a non-fermenter. If each of the two clones consists of a mixture of haploid and diploid cells, zygotes would be produced of composition *Gg*,

GGg, *GGgg* and *Ggg*. The types of asci which each of these zygotes could produce are given in table 2. Each of the combinations would give asci with a 2:2 segregation for galactose fermentation. In addition, asci derived from *GGg* and *GGgg* cells should exhibit 4:0 and 3:1 ratios, the latter depending on a crossover which places *G* and *g* on the same dyad.

The 1:3 and 0:4 ratios are to be expected from the *Ggg* clone. The 1:3 ratio would be obtained as a consequence of the movement of two *G-g* dyads to the same pole following crossing over. To account for the 0:4 ratio it is necessary only to assume that in yeast, as in many other organisms that have been investigated, chromosomal loss in meiosis is the concomitant of the polyploid condition. The loss of the chromosome carrying *G* would result in an ascus in which all four spores would produce nonfermenter clones.

TABLE 2

TYPES OF ASCI EXPECTED FROM A CROSS BETWEEN A GALACTOSE-FERMENTER CLONE AND A NON-FERMENTER ON THE ASSUMPTION THAT EACH CLONE CONTAINS BOTH HAPLOID AND DIPLOID CELLS. THE EFFECT OF CHROMOSOMAL LOSS IS SHOWN ONLY FOR THE *Ggg* GENOTYPE

ZYGOTIC GENOTYPE	TYPES OF ASCI				PHENOTYPIC RATIO WITHIN ASCUS	REMARKS
Gg	*G*	*G*	*g*	*g*	2:2	
GGgg	*Gg*	*Gg*	*Gg*	*Gg*	4:0	
	GG	*GG*	*gg*	*gg*	2:2	
	GG	*Gg*	*Gg*	*gg*	3:1	Crossover
GGg	*Gg*	*Gg*	*G*	*G*	4:0	
	GG	*GG*	*g*	*g*	2:2	
	GG	*Gg*	*G*	*g*	3:1	Crossover
	GG	*G*	*G*	*gg*	3:1	Crossover
Ggg	*Gg*	*Gg*	*g*	*g*	2:2	
	G	*G*	*gg*	*gg*	2:2	
	Gg	*G*	*gg*	*g*	2:2	Crossover
	GG	*gg*	*g*	*g*	1:3	Crossover
	g	*g*	*g*	*g*	0:4	Loss of chromosome carrying *G*

A polyploidy hypothesis can thus provide a formal basis for the interpretation of all of the irregular ratios that have been encountered in crosses. Moreover, the hypothesis is subject to test since it predicts a number of types of identifiable asci, some of which are given in table 2. The tests must be conducted under conditions which will permit the detection of illegitimate hybridization if it occurs. Unfortunately, the most direct evidence, from chromosomal counts, cannot as yet be obtained for lack of suitable cytological techniques.

There are, however, two aspects of the already published data that bear on this question. First, there is the evidence that an irregular ratio for one character is usually accompanied by an irregular ratio for another within the same ascus. Thus among 48 four-spored asci in which irregular ratios occurred,[1, 4] 35 exhibited an irregularity for more than one character. It is perhaps also significant that the 48 asci were produced from crosses in which one parent was either a clone (ClA), obtained from an ascospore of *Saccharomyces carlsbergensis*, or a descendant of this clone. An interpretation of this information in terms of the gene-conversion hypothesis requires the somewhat difficult assumption that the capacity for conversion can affect several loci simultaneously in one pedigree but not in another; the clone ClA is regarded as a "converter" parent which transmits the capacity for conversion to its progeny.

The same evidence can be explained readily on the basis of polyploidy. The simultaneous occurrence of irregular ratios for two or more characters in an ascus would be expected if the zygote were polyploid. For example, zygotes of composition *GGg FFf* should produce a large proportion of asci with 4:0 ratios for both galactose fermentation and flaky. Moreover, the selection for lines which give a high frequency of irregularity could be interpreted as a selection for the polyploid condition. It would follow from this that the ClA clone and some of its descendants were probably diploid, to take the simplest view.

Second, the most striking and direct support for the polyploidy hypothesis comes from the observations that crosses which yield irregular ratios for other characters also produce mating type disturbances. By analogy with the segregations for *G* and *g* in table 2, we should expect asci in which one, two, or four spores are of composition $a\alpha$. These spores, according to our experience with Ascus 55, should give rise to clones that will not mate with either the a or α standards. Five asci have been reported[1, 4, 6] in which all of the spores gave "sterile" clones. It is noteworthy that each of the five asci, like Ascus 55, exhibited irregular ratios for other characters as well. Mundkur[4] also makes reference to asci in which two of the four spores gave sterile clones. We ourselves have found an ascus in which only one of the spores was of this type. Tests of the clone derived from this spore have shown that the spore was $a\alpha$. The occurrence of sterile clones and their correlation with the $a\alpha$ genotype is convincing evidence of the role of polyploidy in the production of irregular ratios.

The foregoing discussion indicates that the polyploidy hypothesis offers a basis for an interpretation of irregular ratios on conventional grounds and moreover that it is supported by the evidence at hand. Further investigation is required to determine the extent to which other explanations may be necessary.

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[1] Lindegren, C. C., *The Yeast Cell*, Educational Publishers Inc., St. Louis, 1949.

[2] Mundkur, B. D., *Current Science*, **19**, 84–85 (1950).

[3] Winge, O., and Roberts, C., *Nature*, **165**, 157–158 (1950).

[4] Mundkur, B. D., *Ann. Missouri Bot. Gard.*, **36**, 259–280 (1949).

[5] All of the crosses reported in this paper were between clones derived from single ascospores. We are indebted to Dr. C. C. Lindegren for the parent clones. The symbols refer to genes affecting the following characters: a and α, mating type; G and g, galactose fermentation and non-fermentation (within 96 hours), respectively; F and f, flaky and free dispersion, respectively, in liquid medium. The flaky character was particularly clear-cut in DIFCO Yeast Nitrogen Base medium with 1% glucose added.

[6] Mundkur, B. D., and Lindegren, C. C., *Am. J. Bot.*, **36**: 722–727 (1949).

ON THE FUNCTIONAL EQUATIONS OF THE DIRICHLET SERIES DERIVED FROM SIEGEL MODULAR FORMS

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The striking analogy between the formulas

$$\int_0^\infty e^{-tx} x^{s-1} dx = \Gamma(s) t^{-s}, \; t > 0, \; Re(s) > 0, \tag{1a}$$

and

$$\int_{X>0} e^{-tr(TX)} |X|^{s - \left(\frac{n+1}{2}\right)} dX = \pi^{n(n-1)/4} \Gamma(s) \Gamma\left(s - \frac{1}{2}\right) \cdots \times \Gamma\left(s - \frac{n-1}{2}\right) |T|^{-s}, \; Re(s) > \frac{n-1}{2}, \tag{1b}$$

where, in the second formula, the integration is over the space of positive definite $n \times n$ matrices, $|X|$ represents the determinant of X, and T is a positive definite matrix, leads one to conjecture that there should be a theory connecting Siegel modular forms and Dirichlet series possessing certain types of functional equations, similar to that created by Hecke connecting Dirichlet series and modular forms of one variable.

The first step in the construction of such a theory has been taken by Maass,[1] who showed that from modular forms of the second degree one can derive an infinite set of Dirichlet series possessing functional equations. Contrary to what occurs in the Hecke theory, the derivation of the functional equation is no longer a simple preliminary step, but requires some detailed analysis. Since the proof given by Maass is rather complicated,

and applicable only in the case of modular forms of the second degree, we propose to present here a sketch of a simpler proof for the result of Maass, which possesses the merit of being able to be extended to the general case. Complete results will be presented subsequently.

Let $F(Z)$ be a modular form of the second degree, satisfying the equation

$$F((AZ + B)(CZ + D)^{-1}) = |CZ + D|^{k}F(Z) \tag{2}$$

for $(AZ + B)(CZ + D)^{-1}$ an element of the modular group of the second degree, and possessing the Fourier expansion,

$$F(X) = \sum_{T} a(T)e^{2\pi i tr(XT)}, \tag{3}$$

where the summation is over all non-negative semi-integers, and the imaginary part of X is positive definite. If we use the parametric representation,

$$\begin{pmatrix} u(x^2 + y^2)/y & ux/y \\ ux/y & u/y \end{pmatrix}, \; u > 0, \; y > 0, \; -\infty < x < \infty, \tag{4}$$

for a positive definite 2×2 matrix, cf. Maass,[1] two domains of importance below are

$$\begin{aligned} D&: \; 0 < u \leq 1, \; -1/2 \leq x \leq 1/2, \; x^2 + y^2 \geq 1, \\ \Delta&: \; -1/2 \leq x \leq 1/2, \; x^2 + y^2 \geq 1. \end{aligned} \tag{5}$$

Let us decompose $F(X)$ into the sum of the two functions defined by

$$F_1(X) = \sum_{|T|=0} a(T)e^{2\pi i tr(TX)}, \tag{6}$$

where the summation is over all non-negative semi-integers of rank 0 or 1, and

$$F_2(X) = \sum_{T>0} a(T)e^{2\pi i tr(TX)}, \tag{7}$$

where the summation is over all positive semi-integers. Then, cf. Maass,

$$\int_D |X|^{s-1}F_2(iX)dX = \int_{X>0} |X|^{s-1}\left(\sum_{|T|} \frac{a(T)}{\epsilon(T)} e^{-2\pi tr(TX)}\right) dX, \tag{8}$$

where $\epsilon(T)$ is the number of elements of the class of T, and the summation on the right is over all distinct classes of 2×2 positive-definite semi-integers. Using the case $n = 2$ of (1b), we obtain for $Re(s)$ sufficiently large,

$$\begin{aligned} f(s) &= \pi^{1/2}\Gamma(s)\Gamma(s + 1/2)\sum_{|T|} a(T)|T|^{-(s+1/2)} = \\ &\quad \int_D |X|^{s-1}[F(iX) - F_1(iX)]dX \\ &= \int_0^1 u^{s-1}\left(\int_\Delta [F(iX) - F_1(iX)]\right)du, \; (|X| = u). \end{aligned} \tag{9}$$

(Here, and in what follows, we write $\int_\Delta$ for $\int_\Delta dxdy/y^2$.)

At first glance, one is tempted to try the classical method of splitting the right-hand integral into two parts, one over $0 < X \leq 1$, and the other over $1 \leq X$, followed by a change of variable $X = X_1^{-1}$, and an application of the modular relation of (2). The presence of the non-constant term $F_1(iX)$ effectively stymies this approach.

To circumvent this difficulty, we use an alternative method based upon analytic continuation. Let us illustrate this method in its starkest simplicity by deriving the functional equation for the Riemann zeta-function. We have, for $Re(s) > 1$,

$$\Gamma(s)\,\zeta(2s) = \int_0^\infty \left(\sum_{n=1}^\infty e^{-n^2x}\right) x^{s-1}dx. \tag{10}$$

Inverting by means of the Mellin inverse, shifting the line of integration to the left past the two poles at $s = 0$, and $s = 1/2$, taking account of the residues at these points, and then re-inverting, we obtain for $Re(s) < 0$,

$$\Gamma(s)\zeta(2s) = \int_0^\infty \left(\sum_{n=1}^\infty e^{-n^2x} + \frac{1}{2} - \frac{\sqrt{\pi}}{2}\,x^{-1/2}\right) x^{s-1}dx. \tag{11}$$

Now applying the functional equation for the theta function, we obtain the functional equation for $\Gamma(s)\zeta(2s)$. This method assumes that we have already demonstrated the analytic continuability of $\Gamma(s)\zeta(2s)$ and know its meromorphic character. All this can be obtained by relatively crude methods based upon the use of the modular property of the theta function. Similarly, below, we shall assume that the analytic properties we require have already been demonstrated, namely the analytic continuability, and the existence of a finite number of poles.

From (9), we obtain by use of the Mellin inversion formula,

$$\int_\Delta [F(iX) - F_1(iX)] = \frac{1}{2\pi i}\int_C f(s)u^{-s}ds, \tag{12}$$

where C is a straight line, $Re(s) = a > s_0$. Shifting the contour to the left, past all the poles of $f(s)$, finite in number, we have for $Re(s) < s_1 < 0$,

$$\int_\Delta [F(iX) - F_1(iX)] = R(u) + \frac{1}{2\pi i}\int_{C_1} f(s)u^{-s}ds, \tag{13}$$

where $R(u)$ is a function of the type $\sum_{k=1}^{N} a_k u^{-b_k}$, and the integral is $0(u^n)$ as $u \to 0$, $u > 0$, for any $n > -s_1$.

On the other hand, we have, from (2),

$$\begin{aligned}\int_\Delta [F(iX) - F_1(iX)] &= \int_\Delta [u^{-k}F(iX^{-1}) - F_1(iX)] \\ &= u^{-k}\int_\Delta F_1(iX^{-1}) - \int_\Delta F_1(iX) + u^{-k}\int_\Delta F_2(iX^{-1})\end{aligned} \tag{14}$$

Using the inequality $tr(TX) \geq u\sqrt{t_0 t_2}$ for $(x, y) \epsilon \Delta$, where

$$T = \begin{pmatrix} t_0 & t_1 \\ t_1 & t_2 \end{pmatrix} > 0, \tag{15}$$

it is easy to demonstrate that the third integral in (14) is $0(e^{-c/u})$ for some $c > 0$ as $u \to 0$, $u > 0$. If we can show that the first two integrals are of the forms $\sum_{k=1}^{n} c_k u^{-dk}$, it will follow that the sum of the first two integrals must equal $R(u)$. Separating out the constant terms corresponding to $T = 0$, and using the parametric representation of 2×2 semidefinite matrices, we find, using the techniques contained in equations (25) and (78) of Maass' paper that the integrals have the required form. Thus,

$$\int_\Delta \Big(\sum_{T>0} a(T) e^{-2\pi i tr(TX)} \Big) - R(u) = u^{-k} \int_\Delta F_2(iX^{-1}) \tag{16}$$

Hence, using (13), for $Re(s) < s_1$, we have

$$\begin{aligned} f(s) &= \int_0^1 u^{s-1-k} \Big(\int_\Delta F_2(iX^{-1}) \Big) du \\ &= \int_{X>0} \Big[\sum_{|T|} \frac{a(T)}{\epsilon(T)} e^{-tr(TX^{-1})} \Big] |X|^{s-k-1} \end{aligned} \tag{17}$$

The change of variable $X = X_1^{-1}$ and integration term-by-term yields the desired functional equation.

The general series discussed by Maass, involving the automorphic solutions of the equation

$$\left(y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \frac{1}{4} + r^2 \right) u = 0 \tag{18}$$

may be handled similarly.

[1] Maass, H., "Modulformen zweiten Grade und Dirichletreihen," *Math. Ann.*, Bd. 122, 90–108 (1950).

THE STRUCTURE OF ALTERNATIVE DIVISION RINGS

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We shall outline the main features of a paper of like title which will appear elsewhere. Except where the contrary is expressly stated, we assume for the sake of brevity that the ring R: (*a*) is *alternative:* $(xx)y = x(xy)$ and $(yx)x = y(xx)$ for all x, y or R; (*b*) is *not associative;* (*c*) has *no divisors of zero:* if $xy = 0$, at least one of x, y is zero; (*d*) has *characteristic not* 2: if $2x = 0$ then $x = 0$.

A linear function $h(x_1, \ldots, x_n)$ from R to R is called *skew-symmetric* if it is zero when two or more of the arguments $x_1, \ldots, x_n$ are equal. If A_i $(i = 1, \ldots, n)$ are subsets of R, $h(A_1, \ldots, A_n)$ denotes the set of elements $h(a_1, \ldots, a_n)$ with a_i in A_i. The *commutator* $(x, y) = xy - yx$ and the *associator* $(x, y, z) = (xy)z - x(yz)$ are skew-symmetric.[1] We show that the linear function $f(w, x, y, z)$, defined by

$$(wx, y, z) = x(w, y, z) + (x, y, z)w + f(w, x, y, z), \tag{1}$$

is skew-symmetric and satisfies the identities

$$3f(w, x, y, z) = (w, (x, y, z)) - (x, (y, z, w)) + (y, (z, w, x)) - (z, (w, x, y)), \tag{2}$$

$$f(w, x, y, z) = ((w, x), y, z) + (w, x, (y, z)). \tag{3}$$

Formulas (1)–(3) are repeatedly used in the proofs.

The *nucleus* $N = N(R)$ consists of all n in R such that $(n, R, R) = 0$. The *center* $C = C(R)$ consists of all c in R such that $(c, R) = (c, R, R) = 0$.

LEMMA 1. $N(R) = C(R)$.

COROLLARY. $(a, b, R) = 0$ *implies* $(a, b) = 0$.

The proof may be deduced from the following facts: If $(a, b, R) = 0$, then $f(a, b, R, R) = 0$, by (1), and hence $((a, b), R, R) = 0$, by (3); that is, $(a, b) \in N$. If $n, nx \in N$, $n \neq 0$, then $(x, R, R)n = 0$, by two uses of (1), and hence $x \in N$.

LEMMA 2. *If R is generated by three elements x, y, z, then* (*i*) $\alpha x^2 - \beta x + \gamma = 0$ *for* $\alpha, \beta, \gamma \in C(R)$, $\alpha \neq 0$; (*ii*) $(x, y) \neq 0$.

COROLLARY. *In any R (satisfying* (*a*) – (*d*)) $(a, b) = 0$ *implies* $(a, b, R) = 0$.

In (i) we take $\alpha = (x, y, z)^2$, $\beta = (x, y, z)(x^2, y, z)$, $\gamma = (x, xy, z)^2$. Using known identities we find that $(\alpha, x, y) = 0$ and by extensive use of (1), (3) we show that $\alpha(\alpha, x) = 0$. Thus $(\alpha, S) = (\alpha, S, S) = 0$ where $S = x \cup y \cup z$. It is now easy to verify that $\alpha \in C$. From these results we deduce without much computation that C contains $\delta = (x^2, y, z)^2$, $(x + x^2, y, z)^2 = \alpha +$

$\delta + 2\beta$, and (hence) β. Further use of (1), (3) shows that $\gamma \in C$. The proof of (ii) is now trivial. (It is essential for (ii), and also for the next lemma, that characteristic 2 be excluded.)

LEMMA 3. *If a, b are elements of R such that $(a, b) \neq 0$, then k is in $C(R)$ if and only if $(k, a) = (k, b) = 0$.*

By the Corollaries to Lemmas 1, 2, $(x, y) = 0$ if and only if $(x, y, R) = 0$. In any alternative ring R with this property, define K as the set of all k in R such that $(k, a) = (k, b) = 0$, and S, as the set of all s in R such that $(K, s) = 0$. Thus $C \subset K$, $a, b \in S$. By extensive use of (1), (3) we obtain the identity $(K, R, R)(S, S) = 0$. Therefore, under the hypothesis of Lemma 3, $K = C$.

LEMMA 4. *Every element x of R satisfies a quadratic equation $\alpha x^2 - \beta x + \gamma = 0$ with $\alpha, \beta, \gamma \in C(R)$, $\alpha \neq 0$.*

If the element x of R is not in C, then x is not in N, by Lemma 1. Hence $(x, y, z) \neq 0$ for some y, z. By Lemma 2 (i), $\alpha x^2 - \beta x + \gamma = 0$, where, in particular, $(k, x) = (k, y) = 0$ for $k = \alpha, \beta, \gamma$. By Lemma 2 (ii), $(x, y) \neq 0$, and hence, by Lemma 3, $\alpha, \beta, \gamma \in C$. Also $\alpha \neq 0$ by Lemma 2 (i). On the other hand, if x is in C, take $\beta = 2\alpha x$, $\gamma = x^2$ for any non-zero center element α.

In view of Lemma 4, C contains a non-zero element. We define $R//C$ as the set of all formal fractions x/c, $x \in R$, $c \in C$, $c \neq 0$, with: (i) $x/c = y/d$ if and only if $dx = cy$; (ii) $(x/c) + (y/d) = (dx + cy)/(cd)$; (iii) $(x/c)(y/d) = (xy)/(cd)$. Then $R//C$ satisfies our initial hypotheses $(a) - (d)$. Moreover $R//C$ has unit c/c and center $F = C//C$, the *quotient field* of C. We imbed R in $R//C$ by making the identifications $x = (cx)/c$. Clearly $R//C$ is an alternative quadratic algebraic algebra (without divisors of zero) in the sense of Albert.[2] Therefore we may state the main theorem:

THEOREM A. *If R is a not-associative, alternative ring without divisors of zero and of characteristic not 2, then $C(R)$ contains a non-zero element and R can be imbedded in $R//C$, which is an alternative division algebra of order 8 (a so-called Cayley-Dickson algebra) over its center $F = C//C$.*

COROLLARY 1. *If, in addition, C is a field, R is a Cayley-Dickson algebra over C.*

COROLLARY 2. *Every alternative division ring of characteristic not 2 is either (i) a field or skewfield or (ii) a Cayley-Dickson algebra over its center.*

COROLLARY 3. *Every ordered alternative ring is associative.*

For Corollary 3, note that if R is ordered then R has characteristic 0 and no divisors of zero. If $C \neq 0$ then $R//C$ can be ordered by defining $x/c > 0$ if and only if $cx > 0$. If R is not associative we obtain a contradiction from Theorem A. More directly, since every subring of an ordered ring is ordered, we may assume that R satisfies the hypotheses of Lemma 2 and obtain a contradiction by observing that $R//C$ contains a quaternion algebra.

The more detailed paper will also contain important applications of Theorem A to the theory of projective planes. We quote without explanation:

THEOREM B. *If a projective plane π satisfies the Theorem of the Complete Quadrilateral, then some coordinate ring R of π is an alternative division ring of characteristic not 2 (and conversely). Moreover, every coordinate ring of π is isomorphic to R; equivalently, the collineation group of π is transitive on quadrilaterals. And, finally, either π is Desarguesian or R is a Cayley-Dickson algebra over $C(R)$.*

THEOREM C. *A linearly ordered projective plane which satisfies the Theorem of the Complete Quadrilateral is Desarguesian.*

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HAUSDORFF MEASURE AND LEBESGUE AREA

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1. Introduction.—Following the creation of modern integration theory by Lebesgue and Carathéodory, it was natural to extend the concept of area to arbitrary continuous surfaces. This was done in several intuitively satisfactory ways, each offering its own geometric and analytic interest. It is now possible to associate with every surface several numbers, each of which claims a good right to be called the area of the surface, but the relationship between these numbers is far from obvious in the general case. Efforts to determine which equations or inequalities are satisfied by these several areas of a surface have greatly added to our knowledge of significant metric and topological properties of continuous surfaces. Thus the lack of uniqueness in the definition of surface area has inspired the creation of a comparative theory, whose main results are summarized in this note. Some of these results are explicitly or implicitly contained in the literature, while others are published here for the first time.

2. Multiplicity Functions.—Suppose f maps X into Y and $y \in Y$. We may attach various meanings to "the multiplicity with which f takes on the value y." The simplest interpretation is given by $N(f, y)$, which is the number (possibly ∞) of elements of $\{x | f(x) = y\}$.

If a topology is distinguished on X, we define $U(f, y)$ to be the number (possibly ∞) of components of $\{x | f(x) = y\}$; and if X is a manifold with

boundary, such as a closed k-cell, we also consider $V(f, y)$, the number (possibly ∞) of those components of $\{x | f(x) = y\}$ which do not meet the boundary of X.

Furthermore, if X is a compact locally connected subset of a k-dimensional manifold and Y is a k-dimensional Euclidean space, we define $S(f, y)$ and $M(f, y)$ in terms of homology theory as in [6].

3. Hausdorff Areas.—We assume that $k \leq n$ are positive integers, $H_{k,n}$ is k-dimensional Hausdorff measure over n-dimensional Euclidean space (see [5]), and that $C_{k,n}$ is the class of all continuous functions on a fixed closed k-cell into n-space.

Motivated by the idea that the area of a mapping $f \in C_{k,n}$ should be a modification of the measure of the range of f, which weights each value of f with the multiplicity with which it is taken on, we are led to consider

$$\begin{aligned} N^*(f) &= \int N(f, y) dH_{k,n}y, \\ U^*(f) &= \int U(f, y) dH_{k,n}y, \\ V^*(f) &= \int V(f, y) dH_{k,n}y \end{aligned}$$

as the Hausdorff areas corresponding to the multiplicity functions N, U, V.

If $k = n$, the multiplicity functions S and M are meaningful and $H_{k,k}$ is simply k-dimensional Lebesgue measure; in this case we can also define the areas

$$\begin{aligned} S^*(f) &= \int S(f, y) dH_{k,k}y, \\ M^*(f) &= \int M(f, y) dH_{k,k}y. \end{aligned}$$

4. Integralgeometric Areas.—Another plausible method for associating an area with a k-dimensional surface in n-space is to form all the orthogonal projections of the surface into k-space, compute the area of each such projection according to any one of the definitions applicable to the simpler case in which $k = n$, and then define the area of the surface as a suitable linear function of the areas of its projections.

Let $p_{n,k}$ be the function on n-space to k-space which maps each n-tuple $(x_1, x_2, \ldots, x_n)$ onto the k-tuple $(x_1, x_2, \ldots, x_k)$. The most general orthogonal projection of n-space onto k-space is a functional superposition of the form $p_{n,k}R$, where R is an element of the orthogonal group G_n of n-space. Thus the projections of a function $f \in C_{k,n}$ are the mappings $(p_{n,k}Rf) \in C_{k,k}$, with $R \in G_n$. The postulate for orthogonal invariance of the linear operation used to define the area of f leads to integration with respect to the Haar measure φ_n of G_n, normalized so that $\varphi_n(G_n) = 1$. Using the coefficient

$$b_{k,n} = \Gamma(1/2) \cdot \Gamma[(n + 1)/2] \cdot \{\Gamma[k + 1)/2] \cdot \Gamma[(n - k + 1)/2]\}^{-1},$$

we thus associate with $f \in C_{k,n}$ the integralgeometric areas

$$N^{**}(f) = b_{k,n} \cdot \int N^*(p_{n,k}Rf)d\varphi_n R,$$
$$U^{**}(f) = b_{k,n} \cdot \int U^*(p_{n,k}Rf)d\varphi_n R,$$
$$V^{**}(f) = b_{k,n} \cdot \int V^*(p_{n,k}Rf)d\varphi_n R,$$
$$S^{**}(f) = b_{k,n} \cdot \int S^*(p_{n,k}Rf)d\varphi_n R,$$
$$M^{**}(f) = b_{k,n} \cdot \int M^*(p_{n,k}Rf)d\varphi_n R,$$

corresponding to the multiplicity functions N, U, V, S, M.

5. Lebesgue Area.—A mapping $f \in C_{k,n}$ is said to be a polyhedron if and only if the domain of f can be so triangulated that f maps each simplex of the triangulation baricentrically onto a rectilinear simplex in n-space. Every rectilinear simplex in n-space has an elementary k-dimensional measure, in the sense that its k-dimensional Hausdorff measure is computable from its vertices by determinants. When applied to polyhedra, all reasonable definitions of area yield the same value, computable by associating with each simplex of a suitable triangulation the elementary k-dimensional measure of its image in n-space and adding all the numbers so obtained.

The class of polyhedra is dense in $C_{k,n}$, relative to the topology of uniform convergence. For any $f \in C_{k,n}$, we may therefore define $L(f)$, the Lebesgue area of f, as the lower limit of the areas of polyhedra approximating f.

6. Comparison of Areas.—Suppose $f \in C_{k,n}$ and the domain of f is the set of all k-tuples $(x_1, x_2, \ldots, x_k)$ such that $0 \leq x_i \leq 1$ for $i = 1, 2, \ldots, k$. The various areas of f are related as follows:

(1) $N^*(f) \geq U^*(f) \geq V^*(f)$.

(2) $N^{**}(f) \geq U^{**}(f) \geq V^{**}(f) \geq S^{**}(f) \leq M^{**}(f) \leq L(f)$.

(3) If $k = n$, then $N^*(f) = N^{**}(f) \geq U^*(f) = U^{**}(f) \geq V^*(f) = V^{**}(f) \geq S^*(f) = S^{**}(f) \leq M^*(f) = M^{**}(f) = L(f)$.

(4) $N^*(f) \geq N^{**}(f)$, $U^*(f) \geq U^{**}(f)$, $V^*(f) \geq V^{**}(f)$.

(5) If the range of f is $(H_{k,n}, k)$ rectifiable (see [5]), then $N^*(f) = N^{**}(f)$ and $U^*(f) = U^{**}(f)$.

(6) If $N^*(f) = N^{**}(f) < \infty$ or $U^*(f) = U^{**}(f) < \infty$, then the range of f is $(H_{k,n}, k)$ rectifiable.

(7) If $\{y \mid V(f, y) > 0\}$ is $(H_{k,n}, k)$ rectifiable, then $V^*(f) = V^{**}(f)$.

(8) If $V^*(f) = V^{**}(f) < \infty$, then $\{y \mid V(f, y) > 0\}$ is $(H_{k,n}, k)$rectifiable.

(9) If f satisfies a Lipschitz condition, which means that there is a number A such that

$$|f(x) - f(y)| \leq A \cdot |x - y| \text{ for } x, y \in \text{domain } f,$$

then $N^*(f) = U^*(f) = V^*(f) = N^{**}(f) = U^{**}(f) = V^{**}(f) = S^{**}(f) = M^{**}(f) = L(f)$.

(10) If $k = 1$, then $N^*(f) = U^*(f) = V^*(f) = N^{**}(f) = U^{**}(f) = V^{**}(f) = S^{**}(f) = M^{**}(f) = L(f)$.

(11) If $k = 2$, then $S^{**}(f) = M^{**}(f)$.

(12) If $k = 2$ and $L(f) < \infty$, then $M^{**}(f) = L(f)$.

(13) If $k = 2$ and $n = 3$, then $M^{**}(f) = L(f)$.

(14) If $H_{n,n}(\text{range } f) = 0$, then $b_{k,n} \cdot L(f) \leq \binom{n}{k} \cdot M^{**}(f)$.

(15) If $U^{**}(f) < \infty$, then $b_{k,n} \cdot L(f) \leq \binom{n}{k} \cdot M^{**}(f)$.

(16) If $k = 2$ and $H_{n,n}(\{y | V(f, y) > 0\}) = 0$, then $M^{**}(f) = L(f)$.

(17) If $k = 2$ and $V^{**}(f) < \infty$, then $M^{**}(f) = L(f)$.

(18) If $k = 2$, $n = 3$ and f is a non-parametric surface, which means that

$$p_{3,2}[f(x)] = x \qquad \text{for } x \in \text{domain } f,$$

then $N^{*}(f) = U^{*}(f) = V^{*}(f) = N^{**}(f) = U^{**}(f) = V^{**}(f) = S^{**}(f) = M^{**}(f) = L(f)$.

(19) If $k = 2$ and f is a saddle surface, which means that

$$\sup_{x \in A} g[f(x)] = \sup_{x \in B} g[f(x)]$$

whenever A is an open subset of the domain of f, B is the boundary of A and g is a numerically valued linear function on n-space, then $V^{*}(f) = V^{**}(f) = S^{**}(f) = M^{**}(f) = L(f)$.

The inequalities in (1) and (2), except for the last one which involves $L(f)$, follow from the corresponding inequalities for the multiplicity functions.

Since M is a lower semicontinuous function, so are M^{*} and M^{**}, by Fatou's lemma. Furthermore it may be verified directly from the definitions that $M^{**}(f) = L(f)$ whenever f is a polyhedron. These facts imply the last inequality in (2).

In (3), only the equation $M^{*}(f) = L(f)$ is not obvious. It was proved in [6].

The propositions (4), (5), (6), (7) and (8) are obtained by applying the fundamental structure theorems for sets of finite Hausdorff measure (see [5], Theorem 9.7) to the definitions of Hausdorff area.

Reference to [4] and [5] yields the equation $N^{*}(f) = S^{**}(f) = L(f)$ for Lipschitz mappings f. Hence (9) follows from (1), (2) and (4). Furthermore (10) may be derived from (9) using parametrization by arc length.

If $k = n = 2$, then $\{y | S(f, y) < M(f, y)\}$ is countable (see [3]). This implies (11).

The proposition (12) is obtained from [4], [7], [8] and [9]. If $n = 3$ and $M^{**}(f) < \infty$, it follows from [2] that $L(f) < \infty$. Consequently (13) may be deduced from (12). These arguments may be generalized to yield (14), (15), (16) and (17).

A proof of (18) is contained in the last section of [5]. It appears that much of the technique employed there could be generalized to higher dimensions.

Extending certain methods developed in [1], it is possible to show that $L(f) \geq V^*(f)$ for every saddle surface $f \in C_{2,n}$. If in addition $n = 2$, then $V^*(f) = S^*(f)$, by virtue of (3) and (11).

In attempting to prove (19) for a general saddle surface $f \in C_{2,n}$ one may, in view of (2), assume that $S^{**}(f) < \infty$. For each $R \in G_n$ the projection $p_{n,2}Rf \in C_{2,2}$ is a saddle surface, hence $V^*(p_{n,2}Rf) = S^*(p_{n,2}Rf)$. By integration over G_n, $V^{**}(f) = S^{**}(f) < \infty$. Reference to (17), (11) and (4) completes the proof of (19).

In view of the preceding results, the relations between the various Hausdorff and integralgeometric areas are now well understood. Examples exist to show that we have listed all inequalities which involve these areas and which are valid for arbitrary continuous surfaces. Corresponding to each inequality, we know in addition a geometric criterion of measure theoretic, point set topological or combinatorial nature which characterizes those surfaces for which equality holds.

Among the Hausdorff and integral geometric areas the function M^{**} appears at present to be most closely related to the Lebesgue area L. The functions M^{**} and L agree for all two-dimensional surfaces in three space and, in the absence of counterexamples, one is tempted to conjecture that the equality holds in general. This wish is natural because up to now the theories of Hausdorff areas and of Lebesgue area have been connected only by the intermediate consideration of the integralgeometric area M^{**}. However, future research may produce a new multiplicity function with a corresponding Hausdorff area more directly related to the Lebesgue area.

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ON THE STRUCTURE OF EXTREMAL POLYNOMIALS

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I. Introductory.—1. The concept of extremal polynomials on a given pointset of the complex plane was introduced and their properties were first studied in a joint paper[1] on the location of the zeros of certain minimum polynomials[2] by M. Fekete and J. von Neumann.

2. The present note's aim is a realization, by one of these authors, of their intention, now some three decades old, to deduce from the definition of the extremal polynomials a *representation* for all extremal polynomials of given degree n, (*a*) when the set S to which the extremal polynomials belong is an *arbitrary* bounded, finite or infinite closed pointset in the complex plane (consisting, if finite, of $n + 1$ points at least); (*b*) when the set S and together with it also the extremal polynomial $p(z)$ to be represented exhibit some properties of *symmetry* (e.g., S is symmetric in the x-axis and $p(z)$ has real coefficients only).

3. The representation afforded here will put into evidence anew two simple (also directly derivable) laws about the distribution of roots of extremal polynomials in relation to the set S to which these polynomials belong—laws which not only remind us of the classical propositions of Lucas and Jensen on the location of the zeros of the derivatives, in relation to the zeros of the original polynomials, but also contain those results as particular cases.[3] Furthermore, the representation enables one to give refinements of the above laws, similar to those obtained by Professor J. L. Walsh, in his most remarkable investigations on critical points,[4] under additional assumptions about some kind of symmetry of the zeros of the given polynomials. This holds good in cases where the assumptions of Walsh do not imply any condition on the *number* of zeros considered, but refer to their *location* only.

4. We shall conclude this introductory by recalling the definitions of the underlying concepts (with their immediate consequences) and by indicating also the connections between extremal polynomials and some minimum polynomials[2] considered by Professor Fejér whose researches suggested our investigations on extremal polynomials and whose discoveries in this field gave the impulse and served for guidance in shaping aims as well as methods.

5. Given a bounded, finite or infinite-closed pointset S of the z-plane and a polynomial $A(z) = z^n + a_1 z^{n-1} + \ldots + a_n$ of degree n with the leading coefficient 1, we call another polynomial $B(z) = z^n + b_1 z^{n-1} + \ldots + b_n$ of the same degree and with the same leading coefficient an *underpolynomial*

of $A(z)$ on S when $|B(z)| < |A(z)|$ for each $z \in S$ except the zeros $z \in S$ of $A(z)$—if such there be—at which also $B(z)$ vanishes.

A polynomial $C(z) = z^n + c_1 z^{n-1} + \ldots + c_n$ is called an *extremal polynomial* on S if it has no underpolynomial $B(z) = z^n + b_1 z^{n-1} + \ldots + b_n$ on S.

6. An immediate consequence of these definitions is that if S is a set of n points, say $z_1, z_2, \ldots, z_n$, then the unique extremal polynomial on S whose degree is n is $\prod_{l=1}^{n}(z - z_l)$. Furthermore, by a result of Fejér ("No root of an extremal polynomial $C(z) = z^n + c_1 z^{n-1} + \ldots c_n$ on S can lie outside the convex hull of S provided that this set consists of at least n points,"), all extremal polynomials of degree n on a convex set S not a single point have the form $\prod_{u=1}^{n}(z - z_u)$ with $z_u \in S$ for $1 \leqslant u \leqslant n$.

7. If S is neither reduced to n points nor is a convex set and if $C(z)$ is an extremal polynomial on S with some of its zero points not in S then $C(z)$ admits a decomposition into two factors, say $C^*(z)$ and $C^{**}(z)$, the first differing from 0 throughout S and the second having all its zero-points in S. Then $C^*(z)$ is an extremal polynomial on S of degree $\leqslant n$.

8. The results of the foregoing two sections clearly indicate that the problem of representation can be *restricted to the case of extremal polynomials* differing from 0 on S.

9. Given a bounded, finite or infinite-closed pointset S of the z-plane, one can associate to every polynomial $A(z) = z^n + a_1 z^{n-1} + \ldots + a_n$, in an arbitrary but definite way, a non-negative finite number $D = D\{A(z);\ S\}$, as a *deviation* of $A(z)$ on S (from 0). Such a deviation is called, in accordance with Fejér, a *monotone* deviation if D decreases when $A(z)$ is replaced by its underpolynomials $B(z)$ (if such there be).[2]

10. *Only extremal polynomials on S can minimize any given monotone deviation D;* thus, the minimum polynomials of such deviations are necessarily extremal polynomials.[5] Conversely, if $C(z) = z^n + c_1 z^{n-1} + \ldots + c_n$ is an extremal polynomial on S, differing from 0 throughout S then $C(z)$ is a minimum polynomial of the (generalized) "Tschebyscheff deviation" $D\{A(z);\ S\} = \underset{z \in S}{\text{Max.}} \left|\frac{A(z)}{C(z)}\right|$, with the weight-function $w(z) = \frac{1}{|C(z)|}$, which is clearly a monotone deviation. Thus the representation of extremal polynomials we are aiming at in the sense of §8 will settle the problem of characterization for all minimum polynomials which differ from 0 throughout S on which the monotone deviation is defined.

II. The Representation of Extremal Polynomials in the General Case.—1. We start this part of our note with the formulation of our (main)

THEOREM I: *Let S be a bounded, finite or infinite pointset of the z-plane, consisting of at least $n + 1$ points when finite and being closed when infinite.*

Let $C(z) = z^n + c_1 z^{n-1} + \ldots + c_n$ *be an extremal polynomial on* S *different from 0 throughout* S. *Then there are* $m + 1$ *points, say* $z_0, z_1, \ldots, z_m$, *in* S *and* $m + 1$ *positive constants, say* $\lambda_0, \lambda_1, \ldots, \lambda_m$ *satisfying the conditions* $n \leqslant m \leqslant 2n$, $\sum_{\mu=0}^{m} \lambda_\mu = 1$ *such that* $C(z)$ *admits the representation:* $C(z) = \frac{\Lambda(z)}{M(z)}$ *where* $\Lambda(z)$ *and* $M(z)$ *are polynomials of the respective orders* m *and* $m - n$ *and with leading coefficients equal to 1, given by the formulas*

$$\Lambda(z) = \sum_{\mu=0}^{m} \lambda_\mu \frac{g(z)}{z - z_\mu}; \quad M(z) = \sum_{\mu=0}^{m} \frac{\lambda_\mu}{C(z_\mu)} \frac{g(z)}{z - z_\mu}; \quad g(z) = \prod_{\mu=0}^{m} (z - z_\mu).$$

2. We rely in the verification of the theorem just formulated upon the following

LEMMA: *Let* S *be a bounded, finite or infinite-closed pointset of the complex* $z = x + iy$ *plane. Let* $p(z) = z^n + p_1 z^{n-1} + \ldots + p_n$ *be a polynomial different from 0 throughout* S. *Let us associate to* S *and* $p(z)$ *a pointset* $\Sigma_{2n} = \Sigma_{2n}(S;\ p(z))$ *of the Euclidean space* E_{2n} *of* $2n$ *dimensions, consisting of the points with the coordinates* $x_1 = R(z^{n-1}p(z))$, $y_1 = I(z^{n-1}p(z))$, $\ldots$, $x_n = R(p(z))$, $y_n = I(p(z))$ *when* z *ranges over the set* S.

A necessary and sufficient condition that $p(z)$ *be an extremal polynomial on* S *is that the convex hull* Γ *of* Σ_{2n} *contains the origin 0 of* E_{2n}.

3. The proof of this lemma can be based on the following fact. Let $p(z) = z^n + p_1 z^{n-1} + \ldots + p_n \neq 0$ for $z \in S$; then $p(z)$ possesses an underpolynomial $q(z) = z^n + q_1 z + \ldots + q_n$ on S if and only if there is a polynomial

$$r(z) = r_1 z^{n-1} + \ldots + r_n \tag{1}$$

of degree $n - 1$ satisfying the inequality

$$|p(z) - r(z)| < |p(z) + r(z)| \quad \text{whenever } z \in S. \tag{2}$$

4. The fact just mentioned is an immediate consequence of the definition of underpolynomials of $p(z)$ on S. Indeed, when

$$|q(z)| < |p(z)| \quad \text{for every } z \in S,$$

then for $r(z) = p(z) - q(z) = (p_1 - q_1)z^{n-1} + \ldots + (p_n - q_n)$ the relations

$$|p(z) + r(z)| = |2p(z) - q(z)| = |p(z)| \left|2 - \frac{q(z)}{p(z)}\right| \geqslant |p(z)| \times \left(2 - \left|\frac{q(z)}{p(z)}\right|\right) > |p(z)| > |q(z)| = |p(z) - r(z)|$$

will clearly hold everywhere on S. Conversely, if there is a polynomial $r(z)$ of the type (1) which satisfies (2) then $q(z) = p(z) - \alpha r(z)$ surely

furnishes an underpolynomial of $p(z)$ on S provided that α satisfies the conditions

$$0 < \alpha \leqq \underset{z \in S}{\text{Min.}} \frac{R\left(\frac{r(z)}{p(z)}\right)}{\left|\frac{r(z)}{p(z)}\right|^2} \tag{3}$$

(Remember: by (2) we have $R\left(\frac{r(z)}{p(z)}\right) > 0$ whenever $z \in S$.) Indeed

$$|p(z) - \alpha r(z)| = |p(z)| \left|1 - \alpha \frac{r(z)}{p(z)}\right| = |p(z)| \left(1 - 2\alpha R\left(\frac{r(z)}{p(z)}\right) + \alpha^2 \left|\frac{r(z)}{p(z)}\right|^2\right)^{1/2} < |p(z)| \text{ if } 0 < \alpha \left|\frac{r(z)}{p(z)}\right|^2 < 2R\left(\frac{r(z)}{p(z)}\right).$$

5. From the fact just established the truth of the lemma easily follows:

1°. The Condition Is Sufficient: For if 0 belongs to Γ, then $p(z)$ *is* an extremal polynomial on S. Otherwise the inequality (2) would hold for some $r(z) = r_1 z^{n-1} + \ldots + r_n$, and consequently, we would have

$$R(p(z))R(r(z)) + I(p(z))I(r(z)) > 0 \text{ whenever } z \in S, \tag{4}$$

which can be written in the form

$$s_1 R(z^{n-1}\overline{p(z)}) + t_1 I(z^{n-1}\overline{p(z)}) + \ldots + s_n R(\overline{p(z)}) + t_n I(\overline{p(z)}) > 0 \tag{5}$$

if we put $r_1 = s_1 - it_1, \ldots, r_n = s_n - it_n$.

Now (5) indicates that the half-space

$$s_1 x_1 + t_1 y_1 + \ldots + s_n x_n + y_n t_n > 0 \tag{6}$$

contains the pointset Σ_{2n} and, therefore, also its convex hull Γ; but its boundary, the hyperplane $s_1 x_1 + t_1 y_1 + \ldots + s_n x_n + t_n y_n = 0$, passes through the origin 0 of E_{2n}. Hence 0 has a positive distance from Γ. Thus the sufficiency of the condition is established.

6. 2°. The Condition Is Also Necessary: Indeed, if 0 is exterior to Γ then there is a hyperplane in E_{2n} which separates 0 from Γ and, therefore, there exists a half-space (6) which contains Γ and, consequently, also Σ_{2n} in its interior; that is, (5) is fulfilled when z ranges over S and thus (4) and (2) also hold when $r_1 = s_1 - it_1, \ldots, r_n = s_n - it_n$ and $r(z)$ is defined by (1). But (2), as we know, implies the existence of underpolynomials $q(z)$ of $p(z)$ on S. Thus the necessity of the condition is established and the proof of our lemma is complete.

7. The gap between this lemma and our main proposition can now be bridged by the following

THEOREM II: *Let S be a pointset as described in the lemma but consisting at least of $n + 2$ points. Let $C(z) = z^n + c_1 z^{n-1} + \ldots + c_n$ an extremal polynomial on S which differs from 0 when z ranges over S. Then there are $m + 1$ points in S, say $z_0, z_1, \ldots, z_m$ with $n \leqslant m \leqslant 2n$, and $m + 1$ positive constants, say $\lambda_0, \lambda_1, \ldots, \lambda_m$ with $\sum_{\mu=0}^{m} \lambda_\mu = 1$ such that*

$$\sum_{\mu=0}^{m} \lambda_\mu \frac{z_\mu^{\alpha}}{C(z_\mu)} = 0 \text{ for } \alpha = 0, 1, \ldots, n-1, \tag{7}$$

and $C(z)$ is an extremal polynomial also on the subset $s = \{z_0, z_1, \ldots, z_m\}$ of S.

8. This theorem is an easy consequence of our lemma when we apply to it Carathéodory's well-known theorem on the convex hull of arbitrary pointsets in a Euclidean space. In fact, by the lemma the convex hull Γ of $\Sigma_{2n}(S, C(z))$ necessarily contains the origin 0 of E_{2n} when $C(z)$ is an extremal polynomial on S, different for 0 throughout S; hence, by Carathéodory's theorem, 0 is a centroid of $m + 1$ well-chosen points of $\Sigma_{2n}(S, C(z))$ with $1 \leqslant m \leqslant 2n$. This fact is equivalent with the existence of $m + 1$ points $z_0, \ldots, z_m$ in S and $m + 1$ positive constants $k_0, \ldots, k_m$ satisfying the equations

$$\sum_{\mu=0}^{m} k_\mu R(z_\mu^{\alpha}\overline{C(z_\mu)}) = \sum_{\mu=0}^{m} k_\mu I(z_\mu^{\alpha}\overline{C(z_\mu)}) = 0 \text{ for } \alpha = 0, 1, \ldots, n-1.$$

Hence

$$\sum_{\mu=0}^{m} k_\mu z_\mu^{\alpha}\overline{C(z_\mu)} = \sum_{\mu=0}^{m} k_\mu |C(z_\mu)|^2 \frac{z_\mu^{\alpha}}{C(z_\mu)} = 0,$$

which is the same as (7) when we put $\lambda_\mu = k_\mu |C(z_\mu)|^2 : \sum_{\mu=0}^{m} k_\mu |C(z_\mu)|^2$. In order to verify $m \geqslant n$ we remark that, by (7), 0 necessarily belongs to the convex hull of the set $\Sigma_{2n}(s, C(z))$ associated with s in the same way that $\Sigma_{2n}(S, C(z))$ is associated with S. Thus, by our lemma, $C(z)$ is an extremal polynomial on s and, therefore, s cannot consist of fewer than $n + 1$ points; otherwise, each polynomial with leading term z^n and divisible by $\prod_{\mu=0}^{m} (z - z_\mu)$ would yield an underpolynomial of $C(z)$ on s_z; this completes the proof.

9. With the aid of this result we can establish the truth of our main theorem: Consider the polynomial

$$N(z) = \sum_{\mu=0}^{m} \frac{\lambda_\mu z_\mu^{n}}{C(z_\mu)} \frac{g(z)}{z - z_\mu} \tag{8}$$

where $z_0, z_1, \ldots, z_m, \lambda_0, \lambda_1, \ldots, \lambda_m$ are the numbers associated with $C(z)$ in the sense of Theorem II and $g(z) = \prod_{\nu=0}^{m} (z - z_0)$. By (7) we clearly have $\sum_{\mu=0}^{m} \frac{\lambda_\mu z_\mu^n}{C(z_\mu)} = 1$, thus $N(z)$ is a polynomial of degree m with leading coefficient 1. A further application of (7) leads to the fact that the polynomial $N(z)$ has the divisor z^n. (Consider the derivatives $N^{(\beta)}(z)$ of $N(z)$ of order $\beta = 0, 1, \ldots, n-1$ and apply (7) in the evaluation of $N^{(\beta)}(0)$). Now, in view of (8) and the divisibility of $N(z)$ by z^n, we obtain

$$\frac{N(z)}{z^n} = \sum_{\mu=0}^{m} \frac{\lambda_\mu}{C(z_\mu)} \frac{g(z)}{z - z_\mu} = z^{m-n} + \ldots = M(z)$$

whence

$$C(z)M(z) = \sum_{\mu=0}^{m} \lambda_\mu \frac{g(z)}{z - z_\mu} = z^m + \ldots = \Lambda(z).$$

This completes the proof of our main theorem.

10. Remark. The polynomial

$$\Lambda(z) = \sum_{\mu=0}^{m} \lambda_\mu \frac{g(z)}{z - z_\mu} = z^m + \ldots \tag{9}$$

because of the conditions $\lambda_\mu > 0$, $\sum_{\mu=0}^{m} \lambda_\mu = 1$ is an extremal polynomial on the set $s = \{z_0, z_1, \ldots, z_m\}$. In fact, the existence of an underpolynomial

$$L(z) = z^m + \ldots = \sum_{\mu=0}^{m} \frac{L(z_\mu)}{g'(z_\mu)} \frac{g(z)}{z - z_\mu}$$

would imply

$$\sum_{\mu=0}^{m} \left| \frac{L(z_\mu)}{g'(z_\mu)} \right| < \sum_{\mu=0}^{m} \left| \frac{\Lambda(z_\mu)}{g'(z_\mu)} \right| = \sum_{\mu=0}^{m} \lambda_\mu = 1$$

in contradiction to

$$1 = \sum_{\mu=0}^{m} \frac{L(z_\mu)}{g'(z_\mu)} = \left| \sum_{\mu=0}^{m} \frac{L(z_\mu)}{g'(z_\mu)} \right| \leqslant \sum_{\mu=0}^{m} \left| \frac{L(z_\mu)}{g'(z_\mu)} \right|.$$

Hence all divisors of $\Lambda(z)$ (among them $M(z)$) are extremal polynomials on s.

11. If $z_0, z_1, \ldots, z_m$ are *not* collinear then the zeros of $\Lambda(z)$ lie *interior* to the polygon which is the convex hull of s. In case $z_0, z_1, \ldots, z_m$ are collinear, the convex hull is reduced to a line-segment σ on which, *between two consecutive points of s, exactly one zero of* $\Lambda(z)$ *will lie*. By a well-chosen linear

transformation of the z-plane σ will be transformed into a segment of the real axis and, thus, $\Lambda(z)$ (and its divisors) into polynomials with *real* coefficients only. Now, under the condition that a set S consists of *real points only* and, therefore, all extremal polynomials $C(z)$ on it have real roots and, consequently, *real coefficients* only, we can improve our Theorem II, namely we can replace in it the inequality $n \leqslant m \leqslant 2n$ by the equality $m = n$ (this is because of the possibility of simplifying our above lemma, substituting for the set $\Sigma_{2n} \subset E_{2n}$ a set in E_n of points with the coordinates $x_1 = z^{n-1}p(z)$, $x_2 = z^{n-2}p(z)$, ..., $x_n = p(z)$ when z ranges over S). The corresponding refinement of our main theorem leads to $N(z) = 1$, $C(z) = \Lambda(z)$ and thus to the more precise result on the distribution of the zeros of $C(z)$ in relation to the points in S as indicated above.

12. The representation of all extremal polynomials $C(z)$ of degree n as described at the beginning of Part II may yield still further information on the location of zeros of those polynomials: *each* theorem on the roots of equations of the type

$$\sum_{\mu=0}^{m} \frac{\lambda_\mu}{z - z_\mu} = 0; \quad \lambda_\mu > 0, \quad \sum_{\mu=0}^{m} \lambda_\mu = 1 \tag{10}$$

combined with our main theorem is eventually a new source for such information; the reader can find an excellent summary of the most interesting facts concerning (10) in "The Geometry of the Zeros of a Polynomial in a Complex Variable" by Professor M. Marden.[6] But a feature of the results so obtainable is that they presuppose the exact knowledge of the points z_μ appearing in (10) which we generally do not possess when the given set S and the extremal polynomial $C(z)$ are *arbitrary*. Only under certain additional assumptions about them, under the condition that they have properties of *symmetry* can we attack the problem with better chance of success. And here we shall rely upon representation of the extremal polynomials of modified type which utilizes such symmetries.

III. Representation of Extremal Polynomials with Symmetry Properties.—1. First we consider extremal polynomials $C(z) = z^n + c_1 z^{n-1} + \ldots + c_n$ on a given set S *symmetric in the x-axis* which are different from 0 on S and have *real coefficients only*. By our main theorem, $C(z)$ is the divisor of a polynomial $\Lambda(z)$ of degree m, $n \leqslant m \leqslant 2n$, given by the formula (9). The subset $s = \{z_0, z_1, \ldots, z_m\}$ of S which occurs in (9) might be, however, non-symmetric in the x-axis and even if it *is* symmetric, it may happen that factors λ_μ which belong to conjugate terms of (9) are different from each other. In this case the fact that $C(z) \mid \Lambda(z)$ cannot be used to regain the already known laws on distribution of zeros of $C(z)$ in the case now considered and to refine them as we have in mind. But we can easily find a polynomial

$$\Lambda^*(z) = \sum_{\mu=0}^{m^*} \lambda_\mu^* \frac{g^*(z)}{z - z_\mu^*} = z^{m^*} + \ldots \tag{11}$$

of degree $m^* \leqslant 4n + 1$, divisible by $C(z)$ where $s^* = \{z_0^*, z_1^*, \ldots, z^*_{m^*}\}$ is a subset of S symmetric in the x-axis; and $\sum_{\mu=0}^{m^*} \lambda_\mu^* = 1$, with $\lambda_\mu^* = \lambda_\nu^* > 0$ when $z_\mu^* = \bar{z}_\nu^*$; finally, $g^*(z) = \prod_{\mu=0}^{m^*} (z - z_\mu^*)$. In fact, the n equations (7) in theorem II hold, under our present assumptions, together with the further n equations:

$$\sum_{\mu=0}^{m} \lambda_\mu \frac{\bar{z}_\mu^\alpha}{C(\bar{z}_\mu)} = 0; \qquad \alpha = 0, 1, \ldots, n - 1;$$

thus also

$$\sum_{\mu=0}^{m} \lambda_\mu \left(\frac{z_\mu^\alpha}{C(z_\mu)} + \frac{\bar{z}_\mu^\alpha}{C(\bar{z}_\mu)} \right) = 0; \qquad \alpha = 0, 1, \ldots, n - 1$$

is valid, leading to

$$\sum_{\mu=0}^{m^*} \lambda_\mu^* \frac{z_\mu^{*\alpha}}{C(z_\mu^*)} = 0; \; 0 \leqslant \alpha \leqslant n - 1, \text{ with } \sum_{\mu=0}^{m^*} \lambda_\mu^* = 1;$$
$$\lambda_\mu^* > 0; \; \lambda_\mu^* = \lambda_\nu^* \text{ when } z_\mu^* = \bar{z}_\nu^* \tag{12}$$

where $s^* = \{z_1^*, z_1^*, \ldots, z^*_{m^*}\}$ consists of all (distinct) points of $\{z_0, \bar{z}_0, \ldots, z_m, \bar{z}_m\}$. We deduce, from (12), the fact $C(z) | \Lambda^*(z)$ by the same arguments which lead from (7), to $C(z) | \Lambda(z)$.

2. This being settled, which means we have a representation for $C(z)$, we derive the properties of the roots of $C(z)$ from the corresponding results concerning $\Lambda^*(z)$. Now the laws of locations for the zeros of $\Lambda^*(z)$ in relation to the subset s^* of S are the same as for the zeros of derivatives of polynomials $L(z)$ with real coefficients with their roots at $z_0^*, z_1^*, \ldots, z^*_{m^*}$ (this is because of the fact that $\Lambda^*(z)$ can be uniformly approximated by polynomials $L(z) = \sum_{\mu=0}^{m^*} \frac{a_\mu}{b} \frac{g^*(z)}{z - z_\mu^*}$ where b and a_μ are natural numbers, satisfying $a_u = a_v$ when $z_u^* = \bar{z}_v^*$). Hence the possibility of extending the theorems of Jensen and its various refinements by Walsh[4] concerning the distribution of zeros of the derivatives $p'(z)$ of polynomials $p(z)$ with real coefficients in relation to the zeros of the original polynomial $p(z)$, to extremal polynomials $C(z)$ with real coefficients on a given pointset S symmetric to the real axis in relation to this set. The reader may replace in those theorems assumptions on the roots of $p(z) = 0$ by assumptions concerning S and the statements about the roots of $p'(z) = 0$ by statements about the zeros of $C(z)$.

3. By arguments similar to the above can the case of symmetry of S and the zeros of the extremal polynomials, both with respect to the origin,

be treated and result about the distribution of roots obtained, analogous to facts concerning zeros of derivatives of polynomials possessing such symmetry properties.[7] We shall explicitly formulate but one result of this kind:

Let S be a pointset symmetric in 0 and lie in the two closed circular regions

$$R_1\colon |z - \alpha| \leqslant r \text{ and } R_2\colon |z + \alpha| \leqslant r$$

where we have $|\alpha| \geqslant 2r$. Then all zeros of all (odd) extremal polynomials on S other than a simple zero at 0, lie in the closed regions R_1 and R_2.

4. Similarly, results on zeros of derivatives valid under conditions of symmetry of higher order[7] with respect to the origin can be extended to extremal polynomials under corresponding symmetry assumptions.

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[1] *Jahresbericht d. Deutschen Math. Vereinigung*, **31**, 125–138 (1922).

[2] *Cp.* L. Fejér, "Uber die Lage der Nullstellen von Polynomen, die aus Minimumforderungen gewisser Art entspringen," *Math. Ann.*, **85**, 41–48 (1922).

[3] *Loc. cit.* reference 1, pp. 135–137.

[4] *The Location of Critical Points of Analytic and Harmonic Functions*, Am. Math. Soc. Colloquium Publications, Vol. 34, 1950, Chapter II.

[5] *Loc. cit.* reference 1, pp. 137–138.

[6] *Am. Math. Soc. Mathematical Surveys*, Number III, Chapters I and II.

[7] *Loc. cit.* reference 4, Chapter III.

ON THE BOUNDARY BEHAVIOR OF MINIMAL SURFACES

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This paper deals with the analytic extension of a minimal surface across its boundary.

LEMMA 1: *Let $U(x, y)$ be harmonic and absolutely bounded in the half plane $y > 0$. Assume $U(x, y)$ to possess*[1] *continuous boundary values $U(x, 0)$ on $y = 0$ which admit limits $U(\infty, 0)$ and $U(-\infty, 0)$. Assume furthermore that the total variation from $x = -\infty$ to $x = \infty$ of $U(x, 0)$, denoted by $T^{\infty}_{-\infty}[U(x, 0)]$, is bounded. Denote similarly by $T^{b}_{a}[U(x, y)]$ the total variation of $U(x, y)$ as x ranges from a to b where a and $b > a$ are any two numbers including $\pm\infty$, and y is kept constant. Then, for $y > 0$,*

$$T^{\infty}_{-\infty}[U(x, y)] \leq T^{\infty}_{-\infty}[U(x, 0)],$$

and

$$\lim_{y \to 0} T^{b}_{a}[U(x, y)] = T^{b}_{a}[U(x, 0)].$$

Proof: $U(x, y) = U(P)$ is for $y > 0$ expressed by the formula

$$U(P) = \pi^{-1} \int_{Q=-\infty}^{\infty} f(Q) d\theta(PQ)$$

where $\theta(PQ)$ is the angle from the negative y-axis to the vector from P to Q, and $f(Q) = U(Q)$ for Q on the x-axis. Integration by parts yields

$$U(P) = -\pi^{-1} \int \theta(PQ) df(Q) + (f(\infty) - f(-\infty))/2. \quad (1)$$

Differentiating with respect to x we find

$$(\partial/\partial x) U(x, y) = \pi^{-1} \int -\frac{\partial}{\partial x} \theta(PQ) df(Q), \quad (2)$$

which we also write with θ_x instead of $(\partial/\partial x)\theta(PQ)$ and where we note that $-\theta_x$ is positive. Thus

$$T_{-\infty}^{\infty}[U(x, y)] = \int_{-\infty}^{\infty} |dU(x, y)| \leq \int \int_{-\infty}^{\infty} -\theta_x \, dx \, |df(Q)| / \pi. \quad (3)$$

Now $\int_{-\infty}^{\infty} -\theta_x \, dx = \pi/2 - (-\pi/2) = \pi$. Hence the desired result

$$T_{-\infty}^{\infty}[U(x, y)] \leq T_{-\infty}^{\infty}[U(x, 0)].$$

Similarly $T_a^b[U(x, y)] \leq \pi^{-1} \int_{-\infty}^{\infty} [\theta(P_1Q) - \theta(P_2Q)] |df(Q)|$, $P_1 = P(a, y)$, $P_2 = P(b, y)$. The bracketed expression is bounded by π and tends uniformly to π as $y \to 0$, for $a + \epsilon \leq Q \leq b - \epsilon$, and to 0 for $Q \geq b + \epsilon$ or for $Q \leq a - \epsilon$ if $\epsilon > 0$ is assigned sufficiently small. For the two missing intervals from $a - \epsilon$ to $a + \epsilon$ and from $b - \epsilon$ to $b + \epsilon$ we gather from the continuity of $f(Q)$ which entails (as is familiar) that of $T_a^x[f(Q)]$, that their contribution to the last integral becomes for all $y > 0$ less than an arbitrary small $\epsilon' > 0$ provided ϵ is sufficiently small. Thus $\overline{\lim_{y \to 0}} \int_{-\infty}^{\infty} [\theta(P_1Q) - \theta(P_2Q)] |df(Q)| \leq \pi \int_a^b + \epsilon^{-\epsilon} |df(Q)| + \pi\epsilon' \leq \pi \int_a^b |df(Q)| + \pi\epsilon'$, whence

$$\overline{\lim_{y \to 0}} T_a^b[U(x, y)] \leqq T_a^b[U(x, 0)].$$

But since also lower semicontinuity holds, i.e.,

$$\underline{\lim_{y \to 0}} T_a^b[U(x, y)] \geq T_{a+\epsilon}^{b-\epsilon}[U(x, 0)]$$

no matter how small ϵ, hence also for $\epsilon = 0$, we have the desired continuity of $T_a^b[U(x, y)]$ as $y \to 0$.

A theorem for the circle analogous to Lemma 1 was proved by Evans.[2]

LEMMA 1′: *If $U_1(x, y)$, $U_2(x, y)$, $U_3(x, y)$ are harmonic and absolutely bounded in $y > 0$ and assume on $y = 0$ continuous boundary values of bounded total variation, then the length $L(y)$ of the curve of coordinates $U_1(x, y)$, $U_2(x, y)$, $U_3(x, y)$ for y fixed and $-\infty < x < \infty$ tends to the length L_0 of the curve $U_1(x, 0)$, $U_2(x, 0)$, $U_3(x, 0) = f_1(Q), f_2(Q), f_3(Q)$, as $y \to 0$.*

Proof: $\pi L(y) = \pi \int_{-\infty}^{\infty} dx \, [(\partial U_1(x, y)/\partial x)^2 + (\partial U_2/\partial x)^2 + (\partial U_3/\partial x)^2]^{1/2}$

$$= \int_{-\infty}^{\infty} dx[(\int_{-\infty}^{\infty} - \theta_x\, df_1(Q))^2 + \ldots]^{1/2}$$
$$\leq \int_{-\infty}^{\infty} dx[(\int_{-\infty}^{\infty} - \theta_x |df_1(Q)|)^2 + \ldots]^{1/2}$$

since $-\theta_x > 0$.

Now

$$[(\int_{-\infty}^{\infty} - \theta_x |df_1(Q)|)^2 + \ldots]^{1/2} \leq \int_{-\infty}^{\infty} - \theta_x[|df_1(Q)|^2 + \ldots]^{1/2}$$

because this formula expresses the fact that a straight line joining the origin to the point with coordinates $\int_{-\infty}^{\infty} - \theta_x|df_1(Q)|$, $\int_{-\infty}^{\infty} - \theta_x|df_2(Q)|$, $\int_{-\infty}^{\infty} - \theta_x|df_3(Q)|$ is the shortest line of this property. Hence

$$\pi L(y) \leq \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} - \theta_x(|df_1|^2 + \ldots)^{1/2}$$

and by Lemma 1,

$$\overline{\lim_{y \to 0}}\, \pi L(y) \leq \lim_{y \to 0} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} - \theta_x(|df_1(Q)|^2 + \ldots)^{1/2} =$$
$$\pi \int_{-\infty}^{\infty} [|df_1(Q)|^2 + \ldots]^{1/2} = \pi L_0.$$

But the semicontinuity of the length requires

$$\lim_{y \to 0} L(y) \geq L_0.$$

Thus $L(y)$ is continuous as $y \to 0$, q. e. d.

LEMMA 2: *Under the hypotheses of Lemma 1, the total variation of U on segments perpendicular to the x-axis is at most $T_{-\infty}^{\infty}[f(Q)]$.*

Proof: It suffices to prove the case when the segment ends on the x-axis, at the origin. Write $U = U_+ + U_-$, where

$$U_+(P) = \pi^{-1} \int_0^{\infty} - \theta(PQ)df(Q) + \text{const.},$$
$$U_- = \pi^{-1} \int_{-\infty}^{0} - \theta(PQ)df(Q) + \text{const.}$$

We have, with $dP = dy > 0$, $d_P\theta(PQ) = \theta_y\, dy < 0$, if $Q > 0$; $\theta_y\, dy > 0$, if $Q < 0$; $|dU_+(P)| \leqq \pi^{-1} \int_0^{\infty} - d_P\theta(PQ)|df(Q)|$, $|dU_-(P)| \leqq \pi^{-1} \int_{-\infty}^{0} d_P\theta(PQ)|df(Q)|$

Thus, integrating over the segment in question,

$$\int |dU_+(P)| \leq \pi^{-1} \int_0^{\infty} \theta(OQ)|df(Q)| = (1/2) \int_0^{\infty} |df(Q)|$$

and similarly

$$\int |dU_-(P)| \leq 1/2 \int_{-\infty}^{0} |df(Q)|.$$

Hence

$$\int |dU(P)| \leq \int |dU_+(P)| + \int |dU_-(P)| \leq (1/2)T_{-\infty}^{\infty}[f(Q)].$$

LEMMA 3: *Let S be a minimal surface of the three-dimensional ξ, η, ζ space, and let S be bounded by a rectifiable Jordan curve C of length L. Take the conformal mapping of S on the upper half plane $y > 0$ of an x, y-plane which establishes a continuous one-to-one correspondence α between C and the*

x-axis[1] *and for which* ξ, η, ζ *become bounded harmonic functions of* x, y *in* $y > 0$ *with boundary values according to* α, *for which in* $y > 0$ *the two relations hold:*

$$\xi_x^2 + \eta_x^2 + \zeta_x^2 = \xi_y^2 + \eta_y^2 + \zeta_y^2,$$

$$\xi_x\xi_y + \eta_x\eta_y + \zeta_x\zeta_y = 0$$

with $\xi_x = (\partial/\partial x)\xi$, ...

Denote by $\xi^*(x, y)$, $\eta^*(x, y)$, $\zeta^*(x, y)$ *conjugate harmonics of* $\xi(x, y)$, η, ζ *and call the surface* S^*, *formed with these functions as coordinates of its points, the conjugate minimal surface. Denote by* $L(y)$ *and* $L^*(y)$ *the length of the curves on* S *and* S^*, *respectively, corresponding to fixed* y *and* $-\infty < x < \infty$. *Then* $L(y) = L^*(y) \to L(0) = L^*(0)$ *as* $y \to 0$.

Proof: If ξ^*, η^*, ζ^* are taken so as to vanish at (0, 0), they become bounded harmonic functions assuming continuous boundary values of bounded total variation on the x-axis. In fact, let $0 < h' < h$, then

$$\begin{aligned} |\xi^*(a, h) - \xi^*(a, h')| \leq \int_{h'}^{h} |\xi_y^*(a, y)|\,dy &\leq \int_{h'}^{h} (\xi_y^{*2} + \eta_y^{*2} + \zeta_y^{*2})^{1/2}\,dy \\ &\leq \int_{h'}^{h} (\xi_x^2 + \eta_x^2 + \zeta_x^2)^{1/2}\,dy \\ &\leq \int_{h'}^{h} (\xi_y^2 + \eta_y^2 + \zeta_y^2)^{1/2}\,dy. \end{aligned}$$

Now by Lemma 2, $\int_0^\infty |\xi_y(a, y)|\,dy \leq T_{-\infty}^{\infty}[\xi(x, 0)] \leq L(0)$.

Hence $\int^\infty |\xi_y^*(a, y)|\,dy$ converges and is $\leq 3L(0)$, thereby insuring that ξ^* tends to a limit as $y \to 0$. Similar facts hold for η^*, ζ^*. That these boundary values of ξ^*, η^*, ζ^* are continuous follows easily from

$$\begin{aligned} |\xi^*(a, 0) - \xi^*(b, 0)| \leq T_a^b[\xi^*(x, 0)] &\leq \liminf_{y \to 0} T_a^b[\xi^*(x, y)] \\ &\leq \lim_{y \to 0} \int_a^b (\xi_x^{*2} + \eta_x^{*2} + \zeta_x^{*2})^{1/2}\,dx \\ &= \lim_{y \to 0} \int_a^b (\xi_x^2 + \eta_x^2 + \zeta_x^2)^{1/2}\,dx \\ &\leq \text{length of } C \text{ from } x = a \text{ to } x = b, \end{aligned}$$

which also shows that the total variation of the boundary values of ξ^* is continuous and remains bounded. Hence Lemma 1′ applies to ξ^*, η^*, ζ^* as well as to ξ, η, ζ; Lemma 3 follows.

LEMMA 3′: *Under the hypotheses of Lemma 3, we have more generally*

$$L_a^b(y) = L_a^{*b}(y) \to L_a^b(0) = L_a^{*b}(0).$$

Proof: Because of semicontinuity the only other possibility could be $L_a^b(0) < \liminf L_a^b(y)$. But then

$$L(0) = L_{-\infty}^{a}(0) + L_a^b(0) + L_b^{\infty}(0) < \liminf L_{-\infty}^{\infty}(y),$$

which contradicts Lemma 1′. Analogous considerations exclude $L_a^{*b}(0) < \liminf L_a^{*b}(y)$, as $y \to 0$ through an arbitrary sequence.

LEMMA 4: *Denote by $L(0, \gamma)$ the length of the boundary curve of the surface with coordinates $\xi \cos \gamma + \xi^* \sin \gamma$, $\eta \cos \gamma + \eta^* \sin \gamma$, $\zeta \cos \gamma + \zeta^* \sin \gamma$. Then $L_a^b(0, \gamma) = L_a^b(0, 0) = L_a^b(0)$.*

Proof: An elementary computation shows $L(y, \gamma) = L(y, 0)$, and the passage $y \to 0$ gives, in view of Lemma 3′, the desired relation.

LEMMA 5: *Denote by s the length on C. Excepting a set of measure zero with respect to s, we have on C*

$$0 = (d\xi/ds)(d\xi^*/ds) + (d\eta/ds)(d\eta^*/ds) + (d\zeta/ds)(d\zeta^*/ds). \quad (4)$$

Proof: Since

$$L_a^b(0, \gamma) = L_a^b(0, 0) = \int_{x=a}^{b} [(d(\xi \cos \gamma + \xi^* \sin \gamma)/ds)^2 + \dots]^{1/2} ds$$

we find by differentiation with respect to γ, setting $\gamma = 0$,

$$0 = \int_{x=a}^{b} [(d\xi/ds)(d\xi^*/ds) + \dots] ds$$

which is the integrated form of Lemma 5, from which it follows by differentiation with respect to $s = s(b)$.

THEOREM: *Under the assumptions of Lemma 3 about S and its coordinates $\xi, \eta, \zeta(x, y)$,* suppose furthermore that the boundary curve contains an analytic arc A which, without loss of generality, may be taken in the form $\eta = \varphi(\xi)$, $\zeta = \psi(\xi)$ with $\varphi(0) = \psi(0) = \varphi'(0) = \psi'(0) = 0$, and with φ and ψ analytic functions of the argument ξ for sufficiently small values of $|\xi|$; and let the origin of the ξ, η, ζ-space correspond to the origin of the x, y-plane. Then $\xi(x, 0)$, $\eta(x, 0)$, $\zeta(x, 0)$ are analytic functions of x at $x = 0$ and can be continued analytically near the origin across $y = 0$.

With the exception, noted by Riemann, of straight line arcs, no other example of an analytic curve has heretofore been shown to possess the property of automatically making the minimal surface analytically extensible across it.[3]

Proof: Set

$$\lambda = (\xi + i\xi^*)/2, \mu = (\eta + i\eta^*)/2, \nu = (\zeta + i\zeta^*)/2, z = x + iy.$$

$\lambda, \mu, \nu(z)$ are analytic functions for $y > 0$, continuous in $y \geqq 0$. If we take the integral $\int |d\lambda| + |d\mu| + |d\nu|$, extended over the x-axis, a parallel to the x-axis, or over the y-axis, it remains by Lemmas 1, 2, 3 bounded above by $6L(0)$. We now define the continuous function $\sigma(x, y) = \int |d\lambda| + |d\mu| + |d\nu|$ by extending the integral from $(0, 0)$ to $(0, y)$ to (x, y) over straight segments. For sufficiently small $h > 0$, the square q of the corners $-h, +h, h + 2ih, -h + 2ih$ has the property that within q, $\sigma(x, y) < \epsilon$ for given $\epsilon > 0$.

Denote by ρ, M two positive numbers such that for the complex quantity Ξ satisfying $|\Xi| \leqq \rho$, we have $\varphi(\Xi)$ and $\psi(\Xi)$ regular and

$$|\varphi'(\Xi)| \leq M|\Xi|, \psi'(\Xi) \leq M|\Xi|.$$

Take ϵ so small that

$$M\epsilon < 1/8 \text{ and } 4\epsilon < \rho.$$

We determine a function $\Xi(z)$ as solution of the differential equation

$$d\Xi = 2[1 + \varphi'^2(\Xi) + \psi'^2(\Xi)]^{-1}\{d\lambda + \varphi'(\Xi)d\mu + \psi'(\Xi)d\nu\} \quad (5)$$

with $\Xi(0) = 0$.[4] Successive approximations give in the usual manner $\Xi(z)$ by integrating over straight segments from 0 to iy to $x + iy$ for all $x + iy = z$ of q. To show this, we observe, calling Ξ_k the kth approximation, that $\Xi_0 \equiv 0$, and

$$|d\Xi_1| \leq 2d\sigma, |\Xi_1| \leq 2\sigma.$$

Furthermore, suppose that $|\Xi_k| \leq 4\sigma$; then, by (5),

$$\begin{aligned} |d\Xi_{k+1}| &\leq 2[1 - 2M^2|\Xi_k|^2]^{-1}\, d\sigma \\ &\leq 2[1 - 2M^2 . 16\sigma^2]^{-1}\, d\sigma \\ &\leq 2[1 - 32(M\epsilon)^2]^{-1}\, d\sigma \\ &\leq 4d\sigma, \end{aligned}$$

and $|\Xi_{k+1}| \leq 4\sigma$.

Thus Ξ_k exists in q for all integral $k \geq 0$. For a certain $M^\wedge$, to be described presently, we find

$$|d|\Xi_{k+2} - \Xi_{k+1}|| \leq |d(\Xi_{k+2} - \Xi_{k+1})| \leq M^\wedge|\Xi_{k+1} - \Xi_k|d\sigma,$$

whence

$$|\Xi_1 - \Xi_0| \leq 2\sigma, |\Xi_2 - \Xi_1| \leq 2M^\wedge\sigma^2/2, |\Xi_{k+1} - \Xi_k| \leq 2\sigma(M^\wedge\sigma)^k/(k+1)!$$

Thus $\Xi_0 + (\Xi_1 - \Xi_0) + (\Xi_2 - \Xi_1) + \ldots$ has the majorant $2(e^{M^\wedge\sigma} - 1)/M^\wedge$ and converges uniformly in q. Here $M^\wedge$ is the maximum of $|G(\kappa) - G(\kappa')| : |\kappa - \kappa'|$ for the following possibilities

$$\begin{aligned} |\kappa|, |\kappa'| \leq \rho, \kappa' \neq \kappa, G(\kappa) &= 2[1 + \varphi'(\kappa)^2 + \psi'(\kappa)^2]^{-1} = G_1(\kappa), \\ G(\kappa) &= G_1(\kappa)\varphi'(\kappa), \\ G(\kappa) &= G_1(\kappa)\psi'(\kappa). \end{aligned}$$

Now set $\Xi = \lim_{k \to \infty} \Xi_k$. We have in q the integrated form of (5),

$$\Xi = 2\int [1 + \varphi'^2(\Xi) + \psi'^2(\Xi)]^{-1}\{d\lambda + \varphi'(\Xi)d\mu + \psi'(\Xi)d\nu\} \quad (6)$$

since the Stieltjes integral on the right hand is the limit as $k \to \infty$ of the analogous integral with Ξ replaced by Ξ_k.

That the construction yields an analytic function of z near all $z = iy$, $y > 0$, follows from the fact that (i) there is for arbitrary initial $x_0 + iy_0$, $y_0 > 0$, a solution of (5) which is analytic, according to the classical existence theorem and (ii) that there is no more than one solution of (5). For call $\delta\Xi$ the difference of two solutions of (5). We have

$$|\delta\Xi| \leq M \int |\delta\Xi| d\sigma$$

with $\delta\Xi$ vanishing initially. This inequality yields $\delta\Xi \equiv 0$ in familiar manner.

That the solution Ξ is continuous in the whole of q follows from the same property of the Ξ_k. The latter fact follows from the convergence of the total variations of λ, μ, ν over parallel segments which approach the x-axis normally. It is true that *a priori* (5) will hold on $y = 0$ only except for a set of measure zero relative to the variable s, but the integrated form holds everywhere.

I say that Ξ coincides with $\xi(x, 0)$ for $y = 0$. In fact, we have on $y = 0$ because $\eta = \varphi(\xi)$, $\zeta = \psi(\xi)$, that almost everywhere in the sense mentioned, in view of (4),

$$2[1 + \varphi'^2(\xi) + \psi'^2(\xi)]^{-1}\{d\xi/2 + \varphi'(\xi d\eta/2 + \psi'(\xi)\, d\zeta/2 + i(d\xi^*/2 + \varphi'(\xi) d\eta^*/2 + \psi'(\xi) d\zeta^*/2)\} = d\xi.$$

Moreover $\xi(0, 0) = 0$. Hence ξ is a possible solution of (5) on $y = 0$ and the solution is unique as proved.

Let $z = x + iy$ be a point such that $z = x - iy$ lies in q. Put $\lambda(z) = \overline{\Xi(\bar{z}) - \lambda(\bar{z})}$.

On $y = 0$, this new λ coincides with the former definition. In fact,

$$\lambda(x) = \overline{\Xi(x) - \lambda(x)} = \xi(x, 0) - (\xi(x, 0) - i\xi^*(x, 0))/2 = (\xi(x, 0) + i\xi^*(x, 0))/2.$$

This new $\lambda(z)$ is regular, being the result of a succession of the two anticonformal mappings $z \to \bar{z}$ and $\bar{z} \to \overline{\Xi(\bar{z}) - \lambda(\bar{z})}$. We can therefore apply the well-known theorem that λ is regular within q plus its image by reflection on $y = 0$, in particular on $y = 0$.

μ and ν can equally be continued across $y = 0$, thus establishing their analyticity at $z = 0$. We effect the extension by putting for $y < 0$, $z = x + iy$,

$$\mu(z) = -\overline{\mu(\bar{z})} + \overline{\varphi(\Xi(\bar{z}))}, \nu(z) = -\overline{\nu(\bar{z})} + \overline{\psi(\Xi(\bar{z}))}.$$

First of all, these new definitions agree with the previous ones for μ, ν on $y = 0$, since there

$$-\overline{\mu(x)} + \overline{\varphi(\Xi(x))} = -(\eta - i\eta^*)/2 + \varphi(\xi) = (\eta + i\eta^*)/2 = \mu(x)$$

and similarly

$$-\overline{\nu(x)} + \overline{\psi(\Xi(x))} = \nu(x).$$

Secondly, the same argument as before yields the fact that for $y < 0$, μ and ν are regular functions of z, thus becoming regular also on the x-axis.

The Theorem is proved.

The idea underlying the analytic extension, of forming a differential equation connecting the extension with the given function, is capable of application in many other situations. We have treated similarly the problem of the "free boundary surface" for minimal surfaces. Other problems concerning analytic boundary conditions come under the same heading. We intend to return to the subject elsewhere.

[1] It suffices to demand assumption of boundary values upon normal approach to the boundary.

[2] Evans, G. C., *The Logarithmic Potential*, Am. Math. Soc. Colloquium Publications VI, New York, 1927, pp. 35ff.

[3] Courant, R., *Dirichlet's Principle, Conformal Mapping and Minimal Surfaces*, Interscience Publishers, 1950, esp. p. 118.

[4] More precisely a solution of the integrated equation (6).

THE CONNECTEDNESS OF THE ORTHOGONAL GROUP IN HILBERT SPACE

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1. A real Euclidean space (of finite dimension number) is orientable and, in particular, its orthogonal transformations form two distinct manifolds, those of the rotations and of the reflections. More than twenty years ago one of us raised the question whether the real Hilbert space, $\mathfrak{R}$, is "non-orientable" in the following sense: If Ω denotes the space of all orthogonal transformations O of $\mathfrak{R}$, then Ω, when topologized by its natural metric (that corresponding to the strong topology of $\mathfrak{R}$) is a connected set. The results of the present note will imply that the answer to this question is in the affirmative, at least if connectedness is meant to be arcwise connectedness.

It is understood that the points O of Ω are meant to be those infinite matrices which are real and unitary (hence bounded), that is, those satisfying both $O'O = E$ and $OO' = E$, where the prime refers to transposition and E denotes the unit matrix. The distance, $|O_1 - O_2|$, between two points of Ω will be meant to be the norm, $|A|$, of the bounded matrix $A = O_1 - O_2$, defined as the least upper bound of the vector length $|Ax|$ when x varies over all vectors x of unit length in $\mathfrak{R}$.

This defines, in particular, the meaning of the statement that a set of points $O = O(t)$, $a \leqq t \leqq b$, in Ω is a continuous path in Ω. The theorem to be proved is that, corresponding to any pair of points O_1, O_2 in Ω, there

exists such a path satisfying $O_1 = O(a)$, $O_2 = O(b)$. In other words, every point of Ω can be connected, by a continuous arc in Ω, with the "origin," E, of Ω.

2. It is well-known that a linear transformation of a real, finite-dimensional, Euclidean space is a rotation (that is, an orthogonal matrix of determinant $+1$) if and only if it is representable in the form e^S, where S is a real, skew-symmetric matrix. Furthermore, it is easily verified that if S is any infinite matrix which is real, bounded and skew-symmetric, then e^S is a point of Ω. It is therefore natural to define a point, O, of Ω to be a rotation, R, of $\mathfrak{R}$ if there exists an $S = -S'$ satisfying $R = e^S$, where S, as every matrix occurring in the sequel, is restricted to be bounded and real. The set of those points O of Ω which are rotations, R, will be denoted by Ω_0. The reflections of $\mathfrak{R}$ can be defined by negation, that is, as those points of Ω which are contained in the complement of Ω_0.

The space Ω_0 of the rotations of $\mathfrak{R}$ is arcwise connected. In fact, if an $R = e^S$ is given, then e^{tS}, where $0 \leqq t \leqq 1$, is a continuous arc in Ω_0 and connects R with $e^{0S} = E$.

In this regard, the rotations of $\mathfrak{R}$ behave as the finite-dimensional rotations. But while the latter form a group, the rotations of $\mathfrak{R}$ fail to do so. In other words, the product of two matrices contained in Ω_0 need not be in Ω_0. This can be seen as follows:

If D_n is any infinite diagonal matrix in which all but a finite number, n, of the diagonal elements are -1 and the remaining n of them are $+1$, then, since the two-rowed matrix

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

is a rotation, it is clear that a D_n is a rotation of $\mathfrak{R}$. Similar considerations show that a $-D_n$ is a rotation if n is even. If n is odd, one will expect that $-D_n$ is a reflection. This happens to be true.[1] In particular, both $D_0 = -E$ and any matrix D_1 are rotations but their product, being a $-D_1$, is a reflection.

3. The fact that the rotations fail to form a group suggests the possibility that perhaps every orthogonal matrix is the product of a finite number of rotations. If this turns out to be true, that is, if every O in Ω can be written in the form $R_1R_2 \ldots R_\mu$, where every R_i is in Ω_0 (the number, $\mu = \mu(O)$, of the factors need not be a bounded function of O on Ω_0), then the arcwise connectedness of Ω follows as a corollary. In fact, the path $O(t) = e^{tS_1}e^{tS_2} \ldots e^{tS_\mu}$, where $0 \leqq t \leqq 1$ and $R_i = e^{S_i}$, will then connect $O(1) = R_1R_2 \ldots R_\mu = O$ with $O(0) = E$.

It turns out that not only do a finite $\mu = \mu(O)$ and the μ rotations $R_i = R_i(O)$ exist for every O but that $\mu(O)$ can be chosen independent of O. In

fact, $\mu(O) = 3$ for every O. In other words, every orthogonal matrix is the product of 3 rotations. It will remain undecided whether the 3 can be reduced to 2.

4. The proof for the sufficiency of $\mu = 3$ will be based on a result of M. H. Martin,[2] which is based on the spectral resolution[3] of an arbitrary unitary matrix, and states that every O can be factored as follows: $O = Qe^S$, where S is skew-symmetric and commutable with Q, and Q is symmetric and $Q^2 = E$ (and hence, Q is orthogonal). Actually, $QS = SQ$ will not be used in the proof of $\mu = 3$.

Since Q is orthogonal, the spectrum of Q must be on the boundary of the unit circle. It follows therefore from the symmetry of Q that the spectrum of Q, being on the real axis, cannot contain any point distinct from ± 1. Hence, Q has no continuous spectrum and is therefore, by the Hilbert-Hellinger theory of orthogonal invariants, orthogonally equivalent to a diagonal matrix. All diagonal elements of the latter are ± 1, of course.

5. If Q is a rotation, then $O = Qe^S$ shows that O is a product of $\mu = 2$ rotations (hence, of $\mu = 3$ rotations as well, since the third can be taken to be E). Consequently, it is sufficient to show that if Q is a reflection, then it is the product of two rotations.

According to the end of Section 4, the matrix Q is orthogonally equivalent to a diagonal matrix, D, in which all diagonal elements are ± 1. Since every orthogonal matrix which is orthogonally equivalent to a rotation is again a rotation (i.e., of the form e^S), it follows that D is a reflection matrix and that Q will be proved to be the product of two rotations if it is shown that D is the product of two rotations. It follows, however, from the rules of Section 2 that D is a $-D_n$ (where, in fact, n is odd) and can therefore be represented as the product of the two rotations D_n and D_0.

[1] Martin, M. H., *Am. J. Math.*, **54**, 592 (1932).
[2] See reference 1, p. 590.
[3] Wintner, A., *Math. Ztschr.*, **30**, 268–277 (1929).

A CLASS OF SOLUBLE DIOPHANTINE EQUATIONS

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1°. Let R be a commutative ring with a unit element, $F(x)$ a homogeneous polynomial of degree n in t indeterminates $x_1, x_2, \ldots, x_t$ with coefficients in R. Let I denote the subring of the coefficients of $F(x)$ in R; that is, the smallest ring containing all of them. We consider the existence of solutions of the diophantine equation

$$F(x) = z^m \tag{1}$$

in R or in I. Here z is an indeterminate and m is a given positive integer.

If $y_1, y_2, \ldots, y_t$ are t new indeterminates and if there exist $t + 1$ polynomials $Q(y)$; $P_i(y)$, $(i = 1, \ldots, t)$, with coefficients in R (or in I) such that

$$F(P(y)) = Q(y)^n \tag{2}$$

identically in the y, (1) will be said to have a t-parameter family of solutions in R (or in I).

2°. THEOREM. *If m is prime to the degree n of $F(x)$, then the diophantine equation (1) always has a t-parameter family of solutions $\mathfrak{M}$ both in R and in I.*

For assume that m is prime to n. If m is less than n, write r for m. Then integers k and l exist uniquely determined by n and r such that

$$kn + 1 = lr, \qquad 0 < k < r, \qquad 0 < l < n.$$

Define polynomials $P(y)$; $Q(y)$ by

$$P_i(y) = y_i F(y)^k, \qquad (i = 1, \ldots, t); \qquad Q(y) = F(y)^l.$$

Then the coefficients of the $P(y)$ and $Q(y)$ lie in I. Since $F(x)$ was assumed to be homogeneous of degree n, (2) holds identically in the y with m equal to r.

If m is greater than n, divide m by n and let the quotient be q and the remainder r. Then if m is prime to n,

$$m = qn + r, \qquad 0 < r < n, \qquad r \text{ prime to } n.$$

With k, l, $P(y)$ and $Q(y)$ as before, let

$$y_i^* = y_i F(y)^k \qquad (i = 1, \ldots, t).$$

Then $F(y^*) = Q(y)^r$. Hence if

$$P_i^*(y) = y_i^* Q(y)^q \qquad (i = 1, \ldots, t)$$
$$Q^*(y) = Q(y),$$

then $F(P^*(y)) = Q^*(y)^m$ identically in the y. Since the polynomials $P^*(y)$ and $Q^*(y)$ have their coefficients in I, the proof is complete.

3°. The most interesting case of this theorem is when I is the ring of ordinary integers. For example the diophantine equation

$$x^n + y^n = z^m$$

has a two parameter family of integral solutions for every m prime to n; the diophantine equation

$$x^4 + y^4 + z^4 = z^m$$

has a three-parameter family of integral solutions for every odd m, and so on. Many other special cases occur in the literature.[1]

4°. The family $\mathfrak{M}$ of solutions of (1) in R consists of vectors $[\xi;\ \eta] = [\xi_1, \xi_2, \ldots, \xi_t;\ \eta]$ of the form

$$\begin{aligned} \xi &= P(\alpha), \quad \eta = Q(\alpha) \qquad m < n, \\ \xi &= P^*(\alpha), \quad \eta = Q^*(\alpha) \qquad m > n. \end{aligned}$$

Here α stands for t arbitrarily chosen elements $\alpha_1, \ldots, \alpha_t$ of R or of I. If the α are such that $F(\alpha) = 0$, we obtain the trivial zero solution of (1) and this is evidently the only solution of the family $\mathfrak{M}$ with $\eta = 0$ if R has zero radical. In any event the solutions of (1) in R with $z = 0$ are entirely independent of the choice of m.

5°. If R is a field, it is easy to show that every solution $[\kappa, \lambda]$ of (1) in R with $\lambda \neq 0$ is of the form

$$\kappa_i = \theta^a \xi_i \ (i = 1, 2, \ldots, t); \ \lambda = \theta^b \eta.$$

Here $[\xi;\ \eta]$ belongs to the family $\mathfrak{M}$, a and b are positive integers depending only on m and n, while θ is a field element depending only on λ. Thus in this case, $\mathfrak{M}$ gives essentially all solutions of (1) with $z \neq 0$.

6°. The situation is quite different for the solutions $\mathfrak{M}$ in I if I is a domain of integrity. $\mathfrak{M}$ by no means exhausts the possible solutions of (1) in I; in fact the components ξ, η of any $\mathfrak{M}$ solution will usually have common factors in I. For example, if I is the ring of integers, the diophantine equation

$$x_1^2 x_2 + x_1 x_2^2 = z^m$$

has a two-parameter family of integral solutions $[\xi_1, \xi_2, \eta]$ for every odd prime m other than three. But the existence of a single integral solution with ξ_1, ξ_2 co-prime [other than the trivial solutions (1, 0; 1), (0, 1; 1)] would disprove Fermat's last theorem.

[1] Dickson, *History of the Theory of Numbers*, Vol. 1.

THE DETERMINATION OF CRYSTAL STRUCTURES FROM X-RAY DATA WITHOUT A KNOWLEDGE OF THE PHASES OF THE FOURIER COEFFICIENTS*

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Introduction.—Since the development by Patterson[1] in 1935 of a method of making use of $|F_{hkl}|^2$ values in the determination of crystal structures, experimenters have been constantly confronted with the interpretation of "Patterson structures." In 1936 Harker[2] pointed out the convenience of investigating certain special sections through three-dimensional Patterson structures depending upon the space group. One of the first attempts to devise a systematic procedure by which the interpretation of Patterson structures might be made easier was published by Wrinch[3] in 1939, using "vector maps." In April, 1950, two papers were read before the American Crystallographic Association which were somewhat related to the work of Wrinch but, from the standpoint of their results, were more related to each other. One of these papers was by Buerger[4] in which "vector sets" were developed. The content[5] of this paper appeared in March, 1950, and the extensions[6] of the method were published in July, 1950. The second paper of this date was by McLachlan[7] who developed the "mixed projections" on the basis of the theory of the behavior of an optical machine built at the University of Utah, for the production of theoretical projections[8] of several kinds. The "mixed projections" are equivalent to the addition of two translated Patterson structures; and the Fourier method of using them was demonstrated. Very recent work has been reported by Garrido[9] and Clastre.[10]

The present paper has two objectives: (*a*) to present the mixed projections in a more concise form, and (*b*) to introduce the multiplication of translated Patterson structures. The multiplication of Patterson structures as contrasted to the addition of them has the advantage that no non-coincidental maxima appear on the final projection, and therefore the crystal structure is deduced without the consideration of ghost atoms.

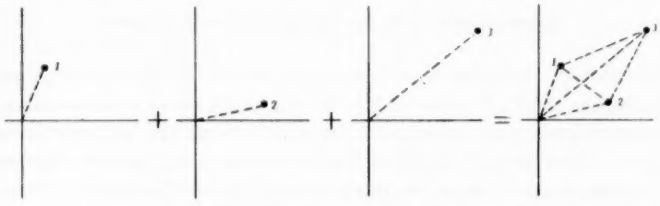
The Additive Property of Scattering Functions.—If a configuration of electrons (such as an atom in a crystal) has a scattering factor $f_1(hkl)$ and is located at the position $x_1y_1z_1 = 000$ in an otherwise empty crystal, then the structure factor of the crystal is $F_{(hkl)} = f_1(hkl)$. If this atom is transferred from the origin to a more general position $x_1y_1z_1$, the new crystal structure factor is obtained from the old one by multiplying it by a factor,

$$e^{2\pi i(hx_1 + ky_1 + lz_1)},$$

i.e.,

$$F_1(hkl) = f_1 e^{2\pi i(hx_1 + ky_1 + lz_1)}.$$

If a second atom is located in a second crystal, a similar expression is used, and so on for the third, fourth, etc. Now, if all these atoms 1, 2, 3 ...



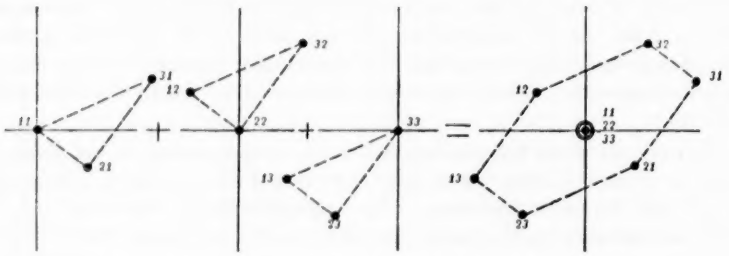
$$f_1 e^{-2\pi i(hx_1 + ky_1)} + f_2 e^{-2\pi i(hx_2 + ky_2)} + f_3 e^{-2\pi i(hx_3 + ky_3)} = F_{hkl}.$$

FIGURE 1

The addition of atoms to form a structure. (***The fact that single atoms may be made equivalent by a shift of origin is being ignored here.***)

were crowded into the same crystal at their respective positions, $x_1y_1z_1$, $x_2y_2z_2$, $x_3y_3z_3$ ..., the final structure factor of the composite crystal is the sum of the structure factors of the separate cells as illustrated in figure 1.

The two facts described in the last paragraph ((1) that a translation of the position of a configuration of electrons in a crystal requires that the



$$f_1 F_h e^{-2\pi i\, hx_1} + f_2 F_h e^{-2\pi i\, hx_2} + f_3 F_h e^{-2\pi i\, hx_3} = f_h F^*_h \text{ or } |F|^2.$$

FIGURE 2

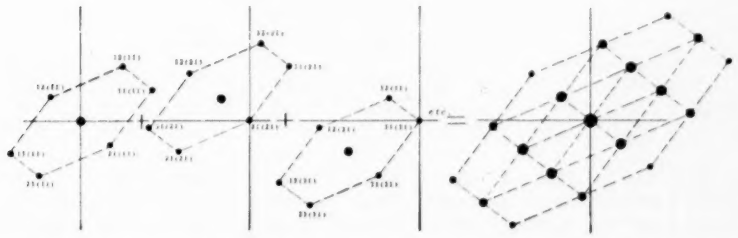
A process for getting Patterson structures by inflating the atoms in real structures and then adding. (***Note that although the equation is in one dimension and the figures are in two dimensions, the principle applies to three dimensions.***)

structure factor must be multiplied by an exponential function of the distance and direction of translation, and (2) that the structure factor of a crystal containing many configurations is the simple sum of the structure

factors of the separate configurations) would be trivial if they were confined to atomic configurations only. However, these two facts are general and apply to the translation and congestion of molecules, groups of molecules or entire cell contents.

The Concept of the "Squared" Crystal.—If, for example, each atom in the final configuration of three atoms 1, 2 and 3 as shown in figure 1 is inflated, so to speak, so that their respective structure factors are f_1f_1, f_1f_2 and f_1f_3, then we have a configuration of "squared" atoms, and the crystal structure factor is f_1F_{hkl}. Then if this configuration of squared atoms is translated so that atom 1 is at the origin (see Fig. 2), the new crystal structure of translated squared atoms is

$$f_1F_{hkl}e^{2\pi i(hx_1 + ky_1 + lz_1)}.$$



$$f_1f_1|F_h|^2e^{-2\pi i\,h(x_1-x_1)} + f_1f_2|F_h|^2e^{-2\pi i\,h(x_2-x_1)} + f_1f_3|F_h|^2e^{-2\pi i\,h(x_3-x_1)}\text{etc.} = |F_h|^4$$

FIGURE 3

Indicating the manner of producing a quadrupled structure from squared structures.

To get a "squared" crystal this process must be repeated as many times as there are atoms, n, in the original structure and all the translated structures collected, without distortion, into one cell. Using this concept, there are n terms in the "squared" crystal structure factor, and, as has been shown by Patterson,[1] the structure factor of the squared crystal is $|F_{hkl}|^2$.

$$\sum_i f_iF_{(hkl)}e^{-2\pi i(hx_i + ky_i + lz_i)} = F_{(hkl)}\,F^*_{(hkl)} = |F_{hkl}|.$$

Extensions Beyond Squared Structures.—If each of the squared atoms in the squared structure is inflated so that its new scattering factor is increased by a factor f_rf_s and translated so that the "atom" rs is at the origin then we have a squared structure of "quadrupled" atoms. Repeating this process (as shown in Fig. 3) as many times as there are squared atoms produces a "quadrupled" structure or a quadrupled crystal, and if it were to exist it would scatter x-rays with an amplitude, $|F_{(hkl)}|^4$. The other crys-

tals such as the tripled crystal will not be discussed here because they do not give fruitful results. In fact, it will be shown that there is no advantage in completing the process of generating a quadrupled crystal because the results desired are found before completion.

A Convenient Symbolism for Treating Structures and Structure Factors.— Let the n atoms in a crystal be numbered serially 1, 2, 3, . . . j . . . n, their positions be designated $x_jy_jz_j$ and their atomic structure factors be f_j. If it is assumed that the knowledge of the serial number of an atom implies a knowledge of its position and scattering factor, then we can abbreviate our treatment considerably. For example, a structure comprising three atoms is simply represented by

$$\begin{vmatrix} 1 \\ 2 \\ 3 \end{vmatrix}.$$

TABLE 1

11(11)	12(11)	13(11)	11(12)	12(12)	13(12)	11(13)	12(13)	13(13)
21(11)	22(11)	23(11)	21(12)	22(12)	23(12)	21(13)	22(13)	23(13)
31(11)	32(11)	33(11)	31(12)	32(12)	33(12)	31(13)	32(13)	33(13)
11(21)	12(21)	13(21)	11(22)	12(22)	13(22)	11(23)	12(23)	13(23)
21(21)	22(21)	23(21)	21(22)	22(22)	23(22)	21(23)	22(23)	23(23)
31(21)	32(21)	33(21)	31(22)	32(22)	33(22)	31(23)	32(23)	33(23)
11(31)	12(31)	13(31)	11(32)	12(32)	13(32)	11(33)	12(33)	13(33)
21(31)	22(31)	23(31)	21(32)	22(32)	23(32)	21(33)	22(33)	23(33)
31(31)	32(31)	33(31)	31(32)	32(32)	33(32)	31(33)	32(33)	33(33)

If an atom j is inflated so that its scattering factor is increased by the factor f_t the new scattering factor is f_jf_t; and if also the atom j is translated a distance $x_j - x_t$, $y_j - y_t$, $z_j - z_t$, then the symbol jt may be used to express both the inflation and the translation. For example, if the entire structure of three atoms 1, 2, and 3 is translated so that atom 1 is at the origin (see Fig. 2) and inflated by f_1 then the new structure is represented by

$$\begin{vmatrix} 1 \\ 2 \\ 3 \end{vmatrix} 1 = 1 \begin{vmatrix} 11 \\ 21 \\ 31 \end{vmatrix}.$$

Following this scheme, a Patterson (or squared crystal) of three points is represented by

$$\begin{vmatrix} 11 & 12 & 13 \\ 21 & 22 & 23 \\ 31 & 32 & 33 \end{vmatrix}.$$

Similarly the appearance of four numbers $jt(rs)$ means a quadrupled atom of scattering factor $f_j f_t f_r f_s$ at the position $(x_j - x_t) - (x_r - x_s)$. For example, if a Patterson structure is inflated by a factor $f_1 f_2$ and translated a distance $-(x_1 - x_2)$ the new structure is represented by

$$\begin{vmatrix} 11 & 12 & 13 \\ 21 & 22 & 23 \\ 31 & 32 & 33 \end{vmatrix} 12 = \begin{vmatrix} 11(12) & 12(12) & 13(12) \\ 21(12) & 22(12) & 23(12) \\ 31(12) & 32(12) & 33(12) \end{vmatrix}.$$

The continuation of this symbolism to represent a quadrupled structure is shown in table 1. Table 1 may be simplified somewhat (see table 2) by application of the following reductions:

$$jt = 0 \quad \text{if} \quad j = t$$

$$jt(rs) = \begin{cases} jt & \text{when } r = s \\ sr & \text{when } j = t \\ jr & \text{when } t = s \\ st & \text{when } j = r \\ 0 & \text{when } j = t = r = s. \end{cases}$$

TABLE 2

0	12	13	21	0	23	31	32	0
21	0	23	21(12)	21	23(12)	21(13)	31	21
31	32	0	31(12)	31	21	31(13)	32(13)	31
12	12(21)	13(21)	0	12	13	32	12(23)	12
0	12	13	21	0	23	31	32	0
32	32(21)	12	31	32	0	31(23)	32(23)	32
13	12(31)	13(31)	23	13	13(32)	0	12	13
23	13	23(31)	21(32)	23	23(32)	21	0	23
0	12	13	21	0	23	31	32	0

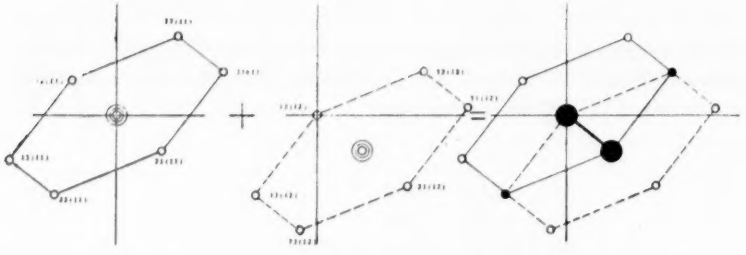
TABLE 3

0	12	13	21	0	23
21	0	23	21(12)	21	23(12)
31	32	0	31(12)	31	21

The Coincidences in Structures.—We notice that a structure of three atoms produces 9 terms in the table of the squared structure and 81 terms in the table of the quadrupled structure. In general, the number of terms in an m-fold structure is n^m, if there are n atoms in the original structure. The squared structure has n coincidences at the origin as represented by the zeros along the diagonal of its table of numbers. The quadrupled structure of three atoms has 15 coincidences at the origin (see table 2) or in general $2n^2 - n$ coincidences. Also in the quadrupled structure there are $n(n - 1)$

non-zero positions of coincidence and each of these occur $n^2 + n - 1$ times, thus producing a Patterson structure of exaggerated atoms as a part of the quadrupled structure.

It is of more value to consider less than an entire quadrupled structure. Let us consider just the coincidences in the first two groups of table 2, as shown in table 3.



$$|F_h|^2 e^{-2\pi i\, h(x_1 - x_1)} + |F_h|^2 e^{-2\pi i\, h(x_1 - x_2)} = |F_h|^2 (1 - e^{-2\pi i\, h(x_1 - x_2)})$$

FIGURE 4

The addition of two Patterson structures to produce coincidences.

The coincidences in table 3 are 0, 21, 31 and 23, and may be shown (see Fig. 4) to be the configuration of the original structure and its inversion as follows:

$$\begin{vmatrix} 0 \\ 21 \\ 31 \\ 23 \end{vmatrix} \quad \bar{1} \quad = \quad \begin{vmatrix} 1 \\ 2 \\ 3 \\ 23(\bar{1}) \end{vmatrix} \qquad \text{or} \qquad \begin{vmatrix} 0 \\ 21 \\ 31 \\ 23 \end{vmatrix} \quad 2 \quad = \quad \begin{vmatrix} \bar{2} \\ \bar{1} \\ 31(2) \\ \bar{3} \end{vmatrix}$$

From figure 4 it may be seen that 0 and 21 are the common points of the structure 1 2 3 and its inversion $\bar{1}\bar{2}\bar{3}$. The coincidences are $2n - 2$ in number.

TABLE 4

0	12	13	21	0	23	31	32	0
21	0	23	21(12)	21	23(12)	21(13)	31	21
31	32	0	31(12)	31	21	31(13)	32(13)	31

The Use of Triple Coincidences.—Three groups of table 2, if properly chosen, produce a set of triple coincidences whose configuration is that of a single original pattern as revealed by table 4.

The choice of the three maxima from a Patterson summation requires some consideration because the third maximum may not be consecutive to the first two.

Special Cases.—When a structure has a center of symmetry, the choice of three maxima in the Patterson summation is not necessary. Provided a through center vector is chosen to combine with the zero maximum, then there is only a single original structure produced.

Table 5 shows the Patterson projection of four atoms $1,2,\bar{1},\bar{2}$ combined with a Patterson projection translated the distance $2\,\bar{2}$. In this case there are just the coincidences $\bar{2}1$, $\bar{2}2$, 12, 0; and adding 2 to these produces the original four numbers: 1, 2, $\bar{1}$ $\bar{2}$.

Special cases that are not so fortunate occur when a vector of length equal to one half the distance through the unit cell are present in the Patterson structure. It can be easily verified by making a table similar to table 5 that such a translation causes ambiguities to appear.

The Multiplication of Squared Structures.—If two Patterson structures are added together after translation, as indicated above, then the single or

TABLE 5

0	12	$1\bar{1}$	$1\bar{2}$	$\bar{2}2$	$12(2\bar{2})$	$1\bar{1}(2\bar{2})$	12
21	0	$2\bar{1}$	$2\bar{2}$	$\bar{2}1$	$\bar{2}2$	12	0
$\bar{1}1$	12	0	21	$\bar{1}1(2\bar{2})$	$1\bar{2}(2\bar{2})$	$\bar{2}2$	$\bar{2}1$
$\bar{2}1$	$\bar{2}2$	12	0	$\bar{2}1(2\bar{2})$	$\bar{2}2(2\bar{2})$	$12(2\bar{2})$	$\bar{2}2$

non-coincident maxima show up as well as the coincidences, because $A(uvw) + 0 = A(uvw)$; but if they are multiplied together the non-coincident maxima do not show up because

$$A(uvw) \times 0 = 0$$

The Fourier Treatment of the Addition Process.—The addition process implies the sum of a Patterson structure represented by $A(uvw)$ and a translated Patterson structure:

$$A(u + \Delta x_{jt}, v + \Delta y_{jt}, w + \Delta z_{jt}).$$

To conserve space the one-dimensional terminology is used

$$\begin{aligned} A(u) + A(u + \Delta x_{jt}) &= \sum_{h=-\infty}^{h=+\infty} |F_h|^2 e^{-2\pi i h u} + \sum_h |F_h|^2 e^{-2\pi i h(u + \Delta x_{jt})} \\ &= \sum_h |F_h|^2 (1 + e^{-2\pi i h \Delta x_{jt}}) e^{-2\pi i h u} \\ &= \sum_h (F_h)_M e^{-\pi i h u} \end{aligned}$$

where

$$(F_h)_M = |F_h|^2 (1 + e^{-2\pi i h \Delta x_{jt}}).$$

Since all Patterson structures are symmetrical it is more convenient to use the reduced equation

$$A(u) + A(u + \Delta x_{jl}) = 2 \sum_{h=0}^{h=\infty} |F_h|^2 [\cos 2\pi hu + \cos 2\pi h(u + \Delta x_{jl})]$$

It is a further convenience to consider that the two cells are translated in opposite directions a distance $^1/_2\Delta x_{jl}$, in which case the useful equation is

$$A(u + {}^1/_2\Delta x_{jl}) + A(u - {}^1/_2\Delta x_{jl}) = 2 \sum_{h=0}^{h=\infty} (F_h)_M \cos 2\pi\, hu$$

where

$$(F_h)_M = 2|F_h|^2 \cos \pi h \Delta x_{jl}.$$

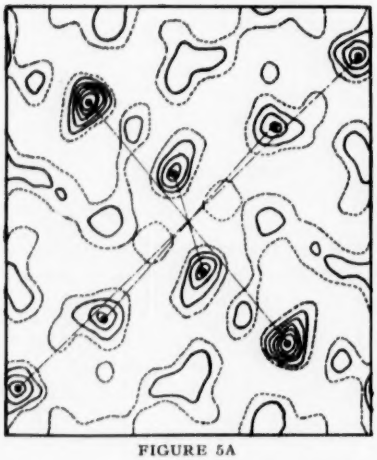


FIGURE 5A

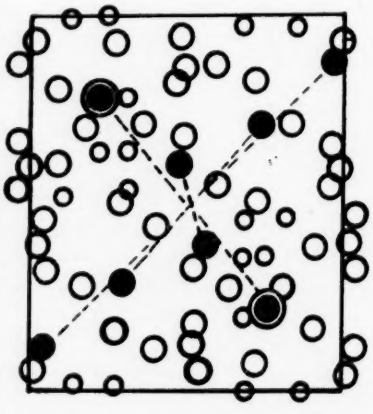


FIGURE 5B

FIGURE 5A. The Fourier projection of hexamethylene-diamine di-hydrobromide by the addition method.

FIGURE 5B. The theoretical projection of hexamethylene di-hydrobromide. Notice how the noncoincident peaks are related to the background in Figure 5A.

This equation implies that if a Patterson summation is made in the usual way and a maximum corresponding to $(\Delta x\, \Delta y\, \Delta z)_{jl}$ is chosen for translation, then the experimental $|F_{hkl}|^2$ values may be multiplied by

$$2 \cos \pi(h \Delta x_{jl} + k \Delta y_{jl} + l \Delta z_{jl})$$

to produce a new set of structure factors $(F_{hkl})_M$. Then the Fourier summation is carried out in the standard manner, thereby solving a structure without a knowledge of the phases of the coefficients, F_{hkl}. Figure 5a shows a

two-dimensional example using the data of Binnie and Robertson[11] from hexamethylenediamine di-hydrobromide. In this projection the bromine atoms are shown in the same positions that the original authors found them. Figure 5b shows the theoretical added projection.

The Fourier Treatment of the Multiplication Process.—The product of two Patterson structures in one dimensions is

$$A(u)A(u + \Delta x_{jl}) = \left(\sum_h |F_h|^2 e^{-2\pi i h u}\right)\left(\sum_{h'} |F_{h'}|^2 e^{-2\pi i' h'(u + \Delta x_{jl})}\right)$$

$$= \sum_h \sum_{h'} |F_h|^2 \text{ and } |F_{h'}|^2 e^{-2\pi i h' \Delta x_{jl}} e^{-2\pi i (h + h')u}$$

$$= \sum_{h + h'} (F_{h + h'})_P e^{-2\pi i (h + h')u}$$

where

$$(F_{h + h'})_P = |F_h|^2 \text{ and } |F_{h'}|^2 e^{-2\pi i h' \Delta x_{jl}}$$

This summation, although comprising the same procedure (after solving for the $(F_{h + h'})_P$ values) as does the usual Fourier summation, involves more terms. Since our machine[12] goes only to $h_{max} = k_{max} = 9$, another, but only approximate, procedure has been adopted. The numbers in a 60×60 Patterson projection are multiplied together at a vector distance indicated by one of the maxima on the Patterson projection. This latter method has very distinct disadvantages because the maxima seldom fall exactly on the divisions. However, since the methods of addition and multiplication are both expressed by Fourier equations, they should be adaptable to computation on such machines as the X-RAC.[13]

Acknowledgments.—The author is indebted to Mr. H. Shimizu for the computation of figure 5a. For encouragement and the use of facilities thanks are given to Dr. John Lewis and Dr. Bronson Stringham, heads of the departments of Metallurgy and Mineralogy. The assistance of Mrs. Alice Morrey Bailey in making the drawings is appreciated.

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*PRIOR LEARNING AS A FACTOR IN SHAPING PERFORMANCE CURVES**

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Communicated by W. R. Miles, November 23, 1950

Introduction.—Performance curves depicting the course of human learning or acquisition of skill have been found, under various experimental circumstances, to be negatively accelerated, S-shaped, linear, hyperbolic, etc. Attempts have been made from time to time to prove that the *true* curve or function is of some particular form. Most commonly perhaps, the "true" function is thought to be negatively accelerated throughout and to have a kinship to the growth function, but strong arguments for functions of other forms, and especially for an S-shaped function, have been presented.

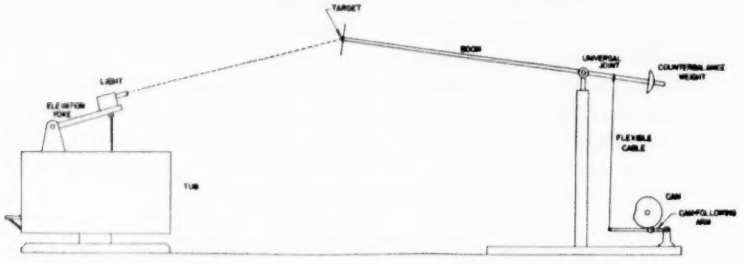

FIGURE 1

A careful consideration of results obtained in a large number of representative studies leads to the conclusion that there is no single "generalized" learning function, no "true" curve of learning or performance. Presumably, the different curves presented by different investigators have not been segments or incomplete parts of some "true" function, but instead, have each reflected the course of learning as determined by many different factors, some of them inseparable from the learning process itself.

McGeoch,[1] in his exhaustive treatment of human learning, lists and discusses six principal determiners of curve-form. They are:

1. Units of measurement.
2. Character of material or activity learned.
3. Characteristics of subjects.
4. Transfer from prior learning.
5. Interference effects and forgetting.
6. Physiological limit.

McGeoch suggests that transfer from prior learning plays a highly significant role in shaping learning curves, but admits that supporting evidence has been generally lacking.

More or less by accident, striking evidence of the role of transfer in shaping performance curves for a complex motor task was recently obtained in an investigation of individual susceptibility to negative transfer effects. This evidence will be summarized here.

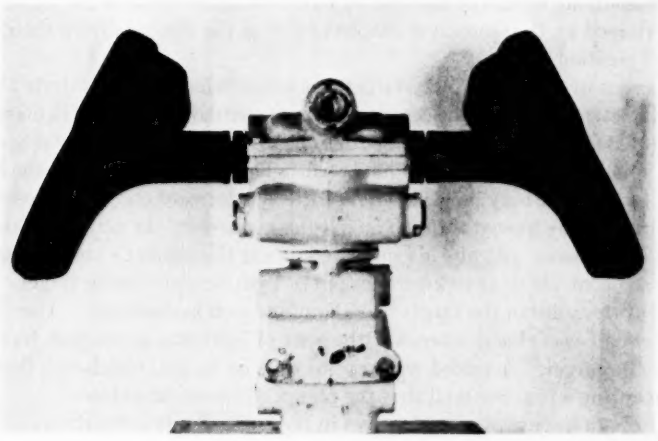

FIGURE 2

The Tasks That Were Learned.—The two motor tasks on which skill was acquired were both available on the Turret Pursuit Apparatus,[2] a hydraulically driven device providing a target which moved along an irregular pathway and a spot of light which subjects learned to keep on the target through a manipulation of manual controls. A schematic drawing of the apparatus is shown in figure 1. A subject, seated in the tub, grasped the two handles shown in figure 2—a close-up view of the control unit. The entire control unit could be turned part of a revolution on a vertical axis, and the handles could be turned a short distance on a horizontal axis. The yoke on the tub carried a light source, as shown in figure 1. A spot of light was projected from this source. The spot, which a subject was to keep on

the target, moved sidewise whenever the tub rotated and vertically whenever the yoke moved up or down.

The yoke was caused to move when the handles on the control unit were turned. With the *standard* arrangement of the controls, the yoke and spot of light moved upward when the handles were pushed downward (thumbs moving toward the body), and moved downward when the handles were pulled upward (thumbs moving away from the body). Rotary (azimuth) movements of the tub arose from the turning of the whole control unit. With the standard control arrangement, pushing ahead on the left handle and pulling backward on the right caused the tub to move in a clockwise direction and the spot of light to the right. Pushing on the right handle and pulling on the left caused a counter-clockwise movement of the tub and a leftward movement of the spot of light. The speed of the tub's azimuth movements and of the yoke's elevation movements (and the accompanying speeds of sidewise and vertical movements of the light) were determined by the amount of displacement of the controls from their mid-point positions.

As seen by the subject, the target was a somewhat darkened circle $1^1/_8$ in. in diameter, located in the center of a white circular plate 12 in. in diameter, mounted on the end of a long boom. The target spot was a circular opening in a small container holding a photocell. The general location of the boom and target plate may be seen in figure 1. The forward end of the boom and the target were moved through an irregular course by the action of a double cam mechanism. As already indicated, it was the subject's task to manipulate the controls so as to keep the spot of light on the moving target. The spot at the point of the target was about one inch in diameter. The photocell circuit was closed whenever the spot of light was more than half-way over the target. A record was kept of time on target, which was the total time during a practice trial that the photocell circuit was closed.

Through the use of special valves in the hydraulic system, it was possible to reverse the action induced by movements of the controls. This gave the so-called *reversed* arrangement of the controls. With the reversed arrangement, the usual movements of the controls gave rise to azimuth movements of the tub and elevation movements of the yoke which were exactly opposite in direction to those prevailing under standard conditions. A subject could be given practice on the apparatus with the controls either standard or reversed, that is, he could acquire skill in performing either the *standard task* or the *reversed task*. Actually, in the studies of individual susceptibility to negative transfer effects, the subjects first acquired skill on the standard task, then received practice on the reversed task, and were tested finally on their retention of skill on the standard task. The marked difference between the performances of the same person on the two tasks led to a study in which a group of subjects first practiced on the reversed task be-

fore learning to perform on the standard task. The shapes of the resulting learning curves were strikingly different, as will now be shown.

Procedures and Results.—Forty male students (Group A) were each given 70 practice trials on the Turret, 10 trials on each of seven consecutive days. Each trial had a duration of 30 seconds and was followed by a rest interval of 30 seconds. A somewhat longer rest of two minutes was given each day between the 5th and 6th trials. For the first 30 trials (Days 1–3), the standard arrangement of the controls was used. In other words, the standard task was learned first. On Days 4–6, for a total of 30 trials, the controls were reversed. This practice constituted the interpolated learning. Ten trials of relearning occurred on Day 7.

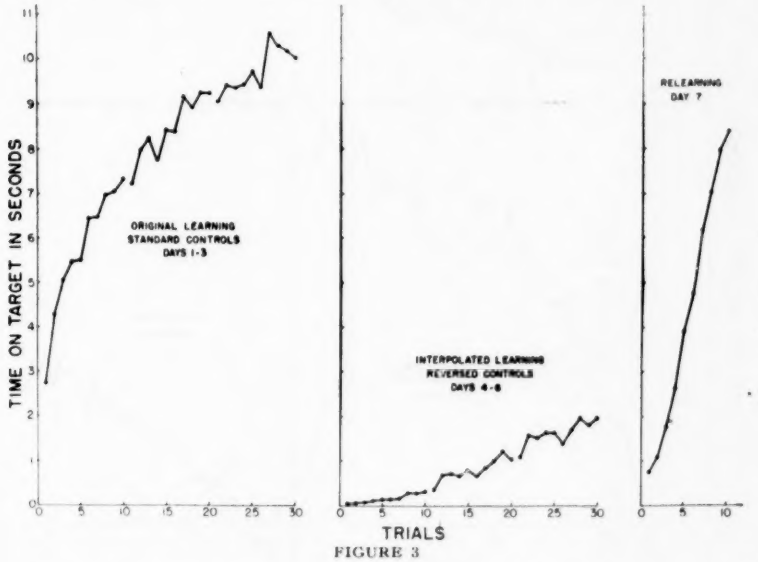

FIGURE 3

Performance data for Group A are summarized in figure 3, where mean time on target in seconds is plotted against trials. There are separate graphs for the three major phases of learning. The curve representing the original learning of the standard task is negatively accelerated and might be regarded as similar, in general form, to many curves obtained under diverse learning conditions. The curve for the learning of the reversed task (middle graph of figure 3) shows a slight tendency to be positively accelerated but is generally linear in form. The relearning curve is predominantly linear but has a slightly S-shaped appearance. It was the marked difference between the curve for the standard task and the curve for the reversed task that led to a second study.

Group B, composed of 14 male students, received a total of 200 practice trials on the Turret, 20 trials on each of ten consecutive days.[3] The length of each trial and of the inter-trial rest interval was 30 seconds, as it had been for Group A. A rest of two minutes was given after each block of five trials. For the first 100 practice trials (Days 1–5), the reversed arrangement of the controls prevailed.[4] The learning of the standard task followed on Days 6–8, for 60 trials in all. Relearning of the reversed task occurred

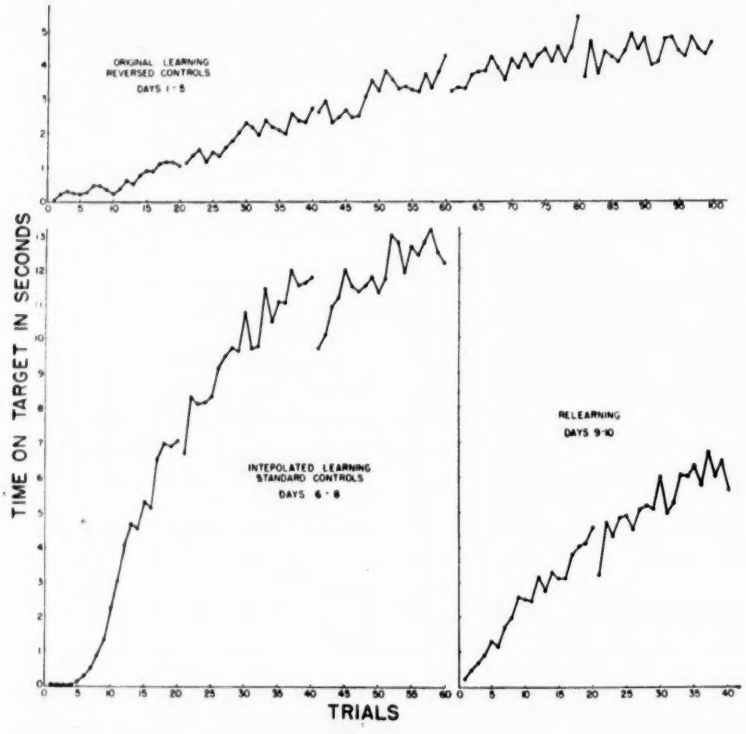

FIGURE 4

on Days 9–10, for 40 trials.

The data are summarized in figure 4 where time on target in seconds is plotted against trials. There are separate graphs for the three phases of learning. The general trend of the performance curve for each phase is apparent.

Discussion.—The curve for original learning in figure 3 should be compared with the curve for interpolated learning in figure 4. Skill in per-

forming the standard task was being acquired in both cases. It is obvious that 100 practice trials on the reversed task had a very pronounced effect on the subsequent learning of the standard task. The curves under the two circumstances of learning were markedly dissimilar. In contrast to the negatively accelerated curve for the learning of the standard task by Group A, the curve for the learning of the same task by Group B has an unmistakable S-shape. That the S-shaped appearance of this latter curve was not an artifact of the grouping of data for individual subjects is shown by figure 5 where the early portions of individual performance curves are

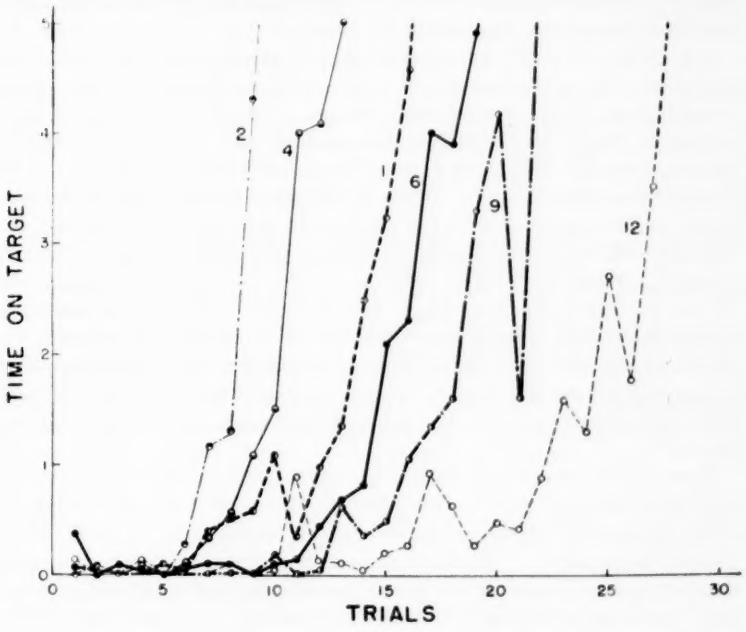

FIGURE 5

presented for six of the 14 subjects. These are representative curves, and all display clear signs of positive acceleration in the learning of the standard task subsequent to the learning of the reversed task.

Attention is called to the discontinuity in the interpolated learning curve for Group B between Trials 40 and 41 (between Days 7 and 8). The sharp drop in proficiency on Trial 41 with relatively slow recovery over several trials may have reflected a "need for warming up," but it may have been related to a persistence of the effects of prior practice on the reversed task. Unfortunately, original learning on the standard task was not continued

through 60 trials for Group A. If it had been, a choice between the two possibilities might have been afforded. As it is, additional data must be obtained to provide the basis for a correct interpretation.

A comparison of the curves for the first 30 trials of the learning of the reversed task by Group B (Fig. 4) and by Group A (Fig. 3) shows them to be quite similar. Their similarity might lead to the conclusion that the prior learning of the standard task had no important bearing on the learning of the reversed task. In this connection, the question must be raised as to why the reversed task was initially far more difficult than the standard task. The greater difficulty may be safely inferred from the over-all level of performance throughout the learning sequence.

The greater difficulty in acquiring skill on the reversed task cannot be explained in terms of any inherent aspect of the task itself, such as greater complexity or higher level of work output. Judged on the basis of complexity or expenditure of effort, the standard and reversed tasks should have been equally difficult to learn. The difference between them must be attributed to prior learning. With the standard arrangement of the controls, the movements required in turning the tub (that is, in moving the light sidewise) were basically similar to those used in steering a car, guiding a bicycle, or turning a boat. The movements employed in changing the vertical position of the spot of light were similar to those most persons have learned in sighting a gun or moving levers on machines. In contrast, the responses required when the controls were reversed were directly opposite to many responses which most persons have long been making. In fact, with the controls reversed, the required movements seemed freakish and bizarre.[5]

There was every reason to believe that initial performance on the reversed task was strongly affected by prior non-laboratory everyday learning. This learning was so great in amount that the responses had become almost implicit in character. The small amount of practice given to Group A on the standard task prior to practice on the reversed task could not have had more than a superficial influence on the old and strongly established habits; all it might have done was slightly reinforced the habits. It is not surprising, therefore, that the performance of the two groups on the reversed task was highly similar. Both groups began the reversed task with old and firmly established habits of responding in ways required for proficient performance on the standard task.

An interesting incidental feature of figure 4 is the performance level attained at the end of relearning. On the 5th day of original learning (of the reversed task), the subjects improved very little and the curve was obviously leveling off at around 4.5 seconds. During relearning (subsequent to 60 trials on the standard task), the reacquisition of skill was quite rapid, the average performance level on Trials 36–40 was better than 6.0

seconds, and at the termination of practice the general trend of the curve was still upward. The higher general level of proficiency on the reversed task following practice on the standard task may have been due to increased motivation; or the subjects may have learned something during interpolated practice which transferred to the reversed task.

Summary.—Two groups of male subjects were given practice on the *standard* and *reversed* tasks provided by the Turret Pursuit Apparatus. The tasks were opposites, that is, they required opposite motor responses to the same pattern of stimuli. One group began practicing on the standard task, went to the reversed task, and then relearned the standard task. The second group began on the reversed task, then practiced the standard task, and relearned the reversed task.

The results showed unmistakably that, under laboratory conditions, transfer from prior learning is an important determiner of the shapes of performance curves. There was good reason for believing that transfer plays a significant role even when the prior learning is of the everyday variety occurring outside of the laboratory.

[*] This work was done under Contract N9onr-93801 between the State University of Iowa and the Special Devices Center, Office of Naval Research.

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[3] Twenty trials per day were given instead of ten (as with Group A) in order to reduce the total number of consecutive days of practice. From a general knowledge of performance on the reversed task, it was predicted that the learning of it would be quite slow.

[4] On Days 1 and 2, for a total of 40 trials, half of the 14 subjects of Group B practiced the reversed task under special verbal instructions while the other half practiced under usual instructions. The special instructions were rather lengthy and provided specific information on how the controls should be turned and how they should not be turned for desired directions of movement of the spot of light. The purpose was to determine whether or not a set to avoid errors and to perform in the correct way could be induced by verbal means. The special instructions were read in full before practice began, and a condensed version was presented at the outset of Day 2. By the end of Day 2, the difference between the performances of the two subgroups was negligible. This led to combining them into the single group for which results are presented in figure 4.

[5] One of the subjects in Group B made a revealing comment at the end of his first day of practice on the reversed task. He had been told that the experiment was related to certain aspects of learning to operate Navy gun turrets. He said, "It's a wonder we ever won the War if we used crazy things like that."

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